Name: ________________________________ **Instructor:** Jeffrey Diller

Math 230: Ordinary Differential Equations
Spring Semester 1999
Final Exam
Friday, May 7

This Examination contains 11 problems on 12 sheets of paper including the front cover. Do all your work on the paper provided and show your computations. When it makes sense to do so, please box your answers—this makes it much easier for me to see what you're doing.

Scores

Question	Possible	Actual
1	10	
2	15	
3	20	
4	15	
5	15	
6	10	
7	15	
8	15	
9	20	
10	20	
11	20	
Total	175	

GOOD LUCK

1. Solve the initial value problem

$$ty' - 3y = (t+1), \qquad y(1) = 2.$$

(10 points)

2. Suppose that $y(t)$ solves the initial value problem

$$y' = \sqrt{x-y}, \qquad y(0) = -2.$$

Use Euler's method with a single step to approximate $y(.1)$. Then give an upper bound for the difference between your approximation and the actual value of $y(.1)$. (15 points)

3. Consider the inhomogeneous differential equation.

$$y'' - 3y' + 2y = 4\sin 2x$$

(a) Find the general solution of the corresponding *homogeneous* equation. (5 points)

(b) Find the general solution of the (inhomogeneous) equation. (10 points)

(c) Convert the inhomogeneous equation into an equivalent 2×2 system of the form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$. (5 points)

4. Suppose that $y(x)$ solves the initial value problem

$$y'' + y' + xy = 0, \qquad y(0) = 2, y'(0) = 0.$$

Using any method you like, find the first three non-zero terms in the power series for y centered at $x = 0$. (15 points)

5. Consider the system of ODE's

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(a) Find the general solution of the system. (10 points)

(b) Use your answer above to write down a fundamental matrix for the system. If you failed to find the answer to the first part, make one up for illustration's sake and get a fundamental matrix from that. **DO NOT** attempt to find the fundamental matrix by exponentiating! (5 points)

6. Suppose that a certain system $\mathbf{x}'(t) = A\mathbf{x}(t)$, of ODE's has a fundamental matrix

$$\Psi(t) = \begin{bmatrix} 2e^{3t} & e^{-t} \\ -3e^{3t} & -2e^{-t} \end{bmatrix}.$$

(a) Write down the general solution of

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

DO NOT evaluate any integrals or multiply any matrices that appear in your answer. (5 points)

(b) Write down the solution of the corresponding homogeneous equation.

(5 points)

7. Consider (but **do not try to solve**) the differential equation

$$y' = (y^2 - 1)(y + 1)^2.$$

(a) Find and classify all equilibrium solutions. (10 points)

(b) On a single plot, sketch the graphs of the solutions y_1, y_2, y_3 satisfying $y_1(0) = 0$, $y_2(0) = 1$, $y_3(0) = 2$. Make sure label to label the graphs! (5 points)

8. Find the critical points of the non-linear system of ODE's

$$\begin{aligned} x' &= (x-2)(y+1) \\ y' &= x - 2y \end{aligned}$$

For one of these critical points (take your pick...), find the corresponding linear system and determine whether the critical point is stable or unstable. (15 points)

9. Existence and uniqueness for first order ODE's.

(a) State the existence and uniqueness theorem for an initial value problem of the form

$$\frac{dy}{dt} = f(y,t), \qquad y(t_0) = y_0.$$

(5 points)

(b) Give an example of such a problem in which one of the hypotheses of the existence and uniqueness theorem fail. (5 points)

(c) How does the conclusion of the theorem improve if $f(x,t) = -p(t)y + q(t)$? (5 points)

(d) Intuitively speaking, what does the existence and uniqueness theorem say if we start plotting solutions $y(t)$ of the ODE. (5 points)

10. Consider the following list of words that we used in class to classify ODEs:

A. autonomous,

B. constant coefficient,

D. homogenous,

E. linear.

For each of the ODE's below, indicate all of the words on the list that accurately describe the ODE. (5 points each)

(a) $\dfrac{dy}{dx} + xy = 3$

(b) $y'' - y^2 = 2$

(c) $y'''(t) = 3y'(t) + 2y(t)$

(d) $\left(\dfrac{d^2y}{dt^2}\right)^2 - x^2y - 2\sin x = 0.$

11. Answer the following questions concerning direction fields and phase portraits. (5 points each)

(a) The following direction field comes from a first order, linear homogeneous 2×2 system $\mathbf{x}' = A\mathbf{x}$. Draw in the eigenvectors for A and say what you can about the corresponding eigenvalues.

(b) Which of the following direction fields comes from a first order autonomous ODE $y' = f(y)$. Why?

(c) The following direction field comes from a first order, non-linear 2×2 autonomous system of ODE's. Identify the critical points of this system and say whether each is stable or unstable.

(d) Consider the non-linear system

$$\begin{aligned} x' &= (x-2)(y-1) \\ y' &= x-2y \end{aligned}$$

from problem 8 on this exam. Note that the point $(x, y) = (1, 0)$ is **NOT** a critical point of the system. What do you expect the phase portrait for this system to look like very close to $(1, 0)$? You can either draw a picture or give a verbal description for your answer.