Math 230, Spring 1999
Mathematica assignment 2
(due Monday, 2/8/99)

The following problems are taken from pages 59 to 67 of the *Differential Equations with Mathematica* book. Please put some effort into presenting your work well. That is, include
comments where appropriate, and use the Format menus to distinguish text from headings
from actual computation. It might work best just to do all the work and then go back and insert comments, format things right, etc. Also, remember that when the instructions don't explicitly require the use of *Mathematica,* you're free to use pencil and paper to do some of the work--whatever gets the job done most effectively.

Problem 2.

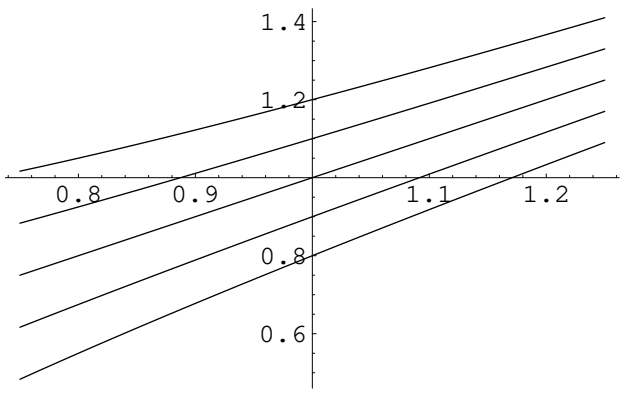
a. (2 points)

```
In[1]:= soln@x_D = First@y@xD . DSolve@8y'@xD + y@xD  x == 2, y@1D == c<, y@xD, xDD
```

$$\text{Out[1]}= \frac{-1 + c + x^2}{x}$$

b. (2 points)

```
In[2]:= Plot@Evaluate@Table@soln@xD, 8c, .8, 1.2, .1<DD, 8x, .75, 1.25<D;
```



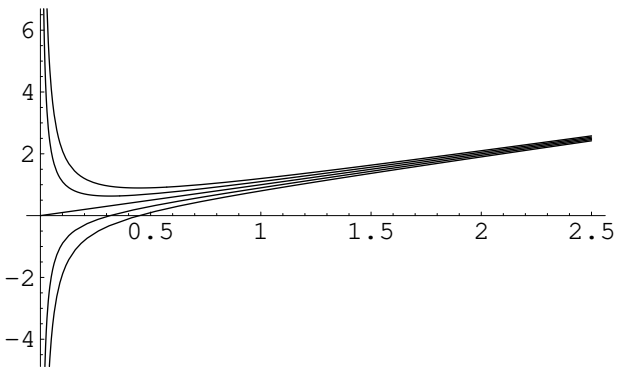
c. (2 points)

```
In[3]:= Table@soln@10^nD, 8n, -2, 1<, 8c, .8, 1.2, .1<D

Out[3]= 88-19.99, -9.99, 0.01, 10.01, 20.01<, 8-1.9, -0.9, 0.1, 1.1, 2.1<,
         80.8, 0.9, 1., 1.1, 1.2<, 89.98, 9.99, 10., 10.01, 10.02<<

In[4]:= %   TableForm

Out[4]//TableForm=
        -19.99      -9.99      0.01      10.01      20.01
        -1.9        -0.9       0.1       1.1        2.1
        0.8         0.9        1.        1.1        1.2
        9.98        9.99       10.       10.01      10.02
```

The columns correspond to c = .8, .9, 1, 1.1, 1.2 going from left to right. The rows correspond to x = 10^n = .01, .1, 1, 10 going from top to bottom.

d. (4 points)

```
Plot@Evaluate@Table@soln@xD, 8c, .8, 1.2, .1<DD, 8x, 0, 2.5<D;
```



All solutions are asymptotic to the same line (looks like y=x) as x goes to infinity−−regardlessof initial condition. Solutions with initial value greater than one tend to infinity as x goes to zero. Those with initial value less than one tend to negative infinity. The solution with initial value one tends to zero as x goes to zero.

Problem 4.

a. (2 points)

```
Clear@yD

DSolve@y'@xD == Hx - Exp@ xDL  Hy@xD + Exp@y@xDDL,
                              y@xD, xD
```

```
Solve::tdep :
 The equations appear to involve the variables to be solved for in an essentially non-algebraic way.
```

```
Solve@-2 a^-x + 2 a^y@xD - x^2 + y@xD^2 == C@1D, 8y@xD<D
```

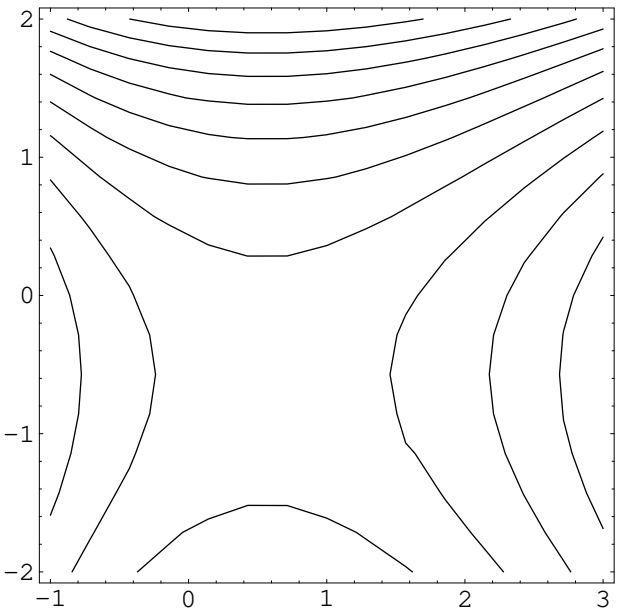
b. (2 points)

I'll cut and paste the left hand side and give it a convenient name (it's important to remove the [x]'s from the y[x]'s):

```
lhs@x_, y_D = SimplifyA 1/2 E^-x H-2 + 2 E^(x+y) - E^x x^2 + E^x y^2LE
```

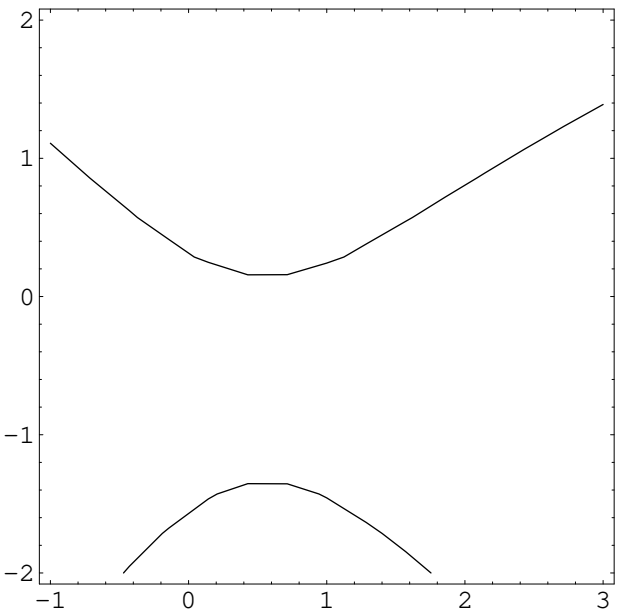
$$-a^{-x} + a^{y} + \frac{1}{2}\left(-x^2 + y^2\right)$$

```
ContourPlot@lhs@x, yD, 8x, -1, 3<, 8y, -2, 2<, ContourShading -> FalseD;
```



c. (3 points)

```
plot1 = ContourPlot@lhs@x, yD, 8x, -1, 3<,
   8y, -2, 2<, Contours -> 8lhs@1.5, .5D<, ContourShading -> FalseD;
```



The top curve is the graph of the solution since it (evidently) passes through the point (1.5,.5).

d. (3 points)

I'll use the previous picture to take a guess at the correct value of y and then use FindRoot to improve the guess:

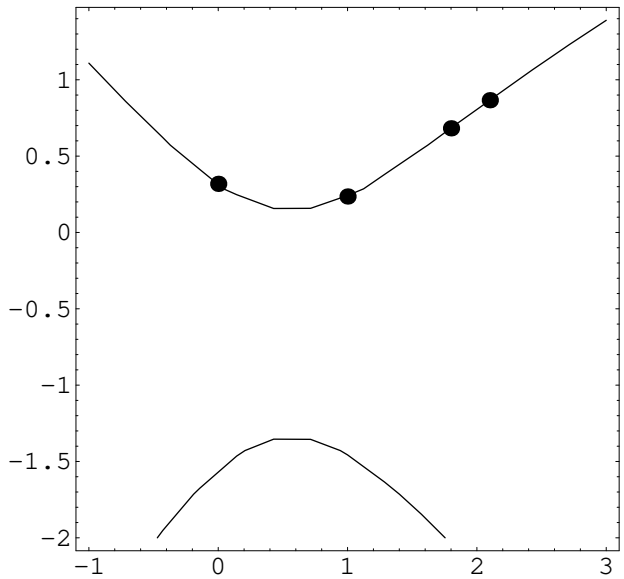
```
FindRoot@lhs@0, yD == lhs@1.5, .5D, 8y, .5<D
FindRoot@lhs@1, yD == lhs@1.5, .5D, 8y, .5<D
FindRoot@lhs@1.8, yD == lhs@1.5, .5D, 8y, .5<D
FindRoot@lhs@2.1, yD == lhs@1.5, .5D, 8y, .5<D
```

```
8y fi 0.318386<

8y fi 0.235633<

8y fi 0.682187<

8y fi 0.866212<
```

That is:

y(0) = .318386...
y(1) = .235633...
y(1.8) = .682187...
y(2.1) = .866212...

```
Show@plot1, Graphics@8PointSize@0.03D, Point@80, .318386<D, Point@81, .235633<D,
                                Point@81.8, .682187<D, Point@82.1, .866212<D<DD;
```

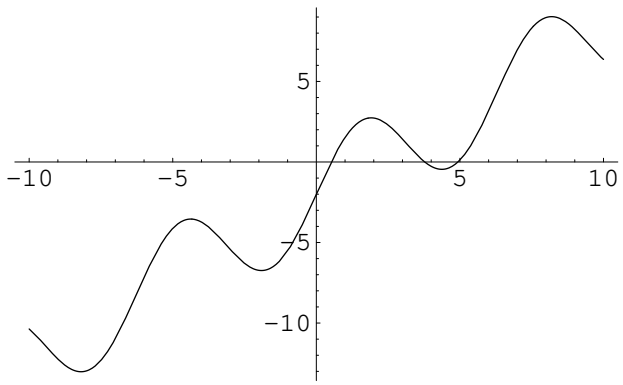


Problem 10.

a. (4 points)

Before I plot the vector field, I'd like to check the graph of the right side of the ODE as a function of y. This will give me some idea of what vertical range to use. Since the equation is autonomous, the horizontal range is unimportant.

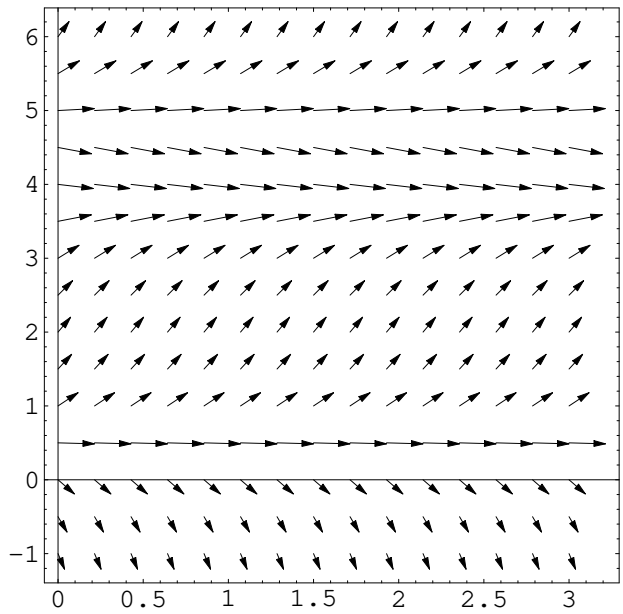
```
In[5]:= Plot@8 Sin@yD + y - 2, 8y, - 10, 10<D;
```



looks like I should expect equilibrium solutions at about y=1/2, y=4 and y=5. Therefore, I'll choose [−1,6]as a range for y.

```
In[6]:= Needs@"Graphics`PlotField`"D
```

```
In[7]:= PlotVectorField@81, 3 Sin@yD + y - 2<, 8t, 0, 3<, 8y, - 1, 6<, Axes -> True,
          ScaleFunction fi H1 &L, Ticks fi None, Frame fi True, AspectRatio fi 1, PlotPoints -> 15D;
```



To find the equilibrium solutions:

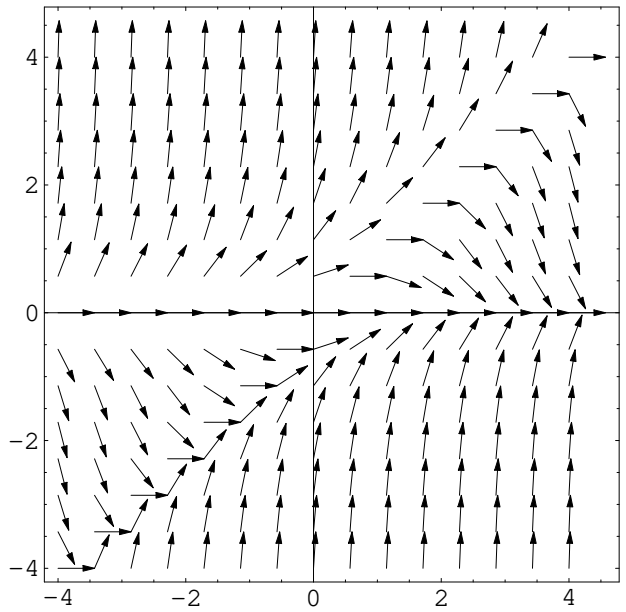
```
In[8]:= FindRoot@8 Sin@yD + y - 2 == 0, 8y, .5<D
        FindRoot@8 Sin@yD + y - 2 == 0, 8y, 4<D
        FindRoot@8 Sin@yD + y - 2 == 0, 8y, 5<D

Out[8]= 8y fi 0.517049<

Out[9]= 8y fi 3.77452<

Out[10]= 8y fi 4.92954<
```

From this and the two plots above, it is apparent that the ODE has two unstable equilibrium solutions: y= 4.9295 and y=.517049 and one stable equilibrium solution: y= 3.77452.

b. (4 points)

This ODE is not autonomous, so I cannot proceed as I did in the last example. I'll have to get along with just the direction field. Note that the right side vanishes (i.e. the direction field is horizontal) when y=0 and y=x.

```
In[11]:= PlotVectorField@81, y^2 - x* y<, 8x, - 4, 4<, 8y, - 4, 4<, Axes -> True,
           ScaleFunction fi H1 &L, Ticks fi None,
         Frame fi True, AspectRatio fi 1, PlotPoints -> 15D;
```



Looks like y=0 is a constant solution of the ODE−−an observation that is easily verified by plugging into the ODE. All solutions y[x] starting below y= 0 appear to converge to y=0 as x goes to infinity. Solutions y[x] starting above y= 0 appear to converge to either infinity or 0. Solutions exhibiting these two types of behavior appear to be separated by a curve asymptotic to the line y=x as x goes to infinity and y=0 as x goes to negative

infinity. At any rate it is clear from the right side of the differential equation that the dividing curve must lie at or above the line y=x, because the right side is negative when y < x (and y > 0) so that solutions lying between y=0 and y=x will slope downward as x becomes large (and positive).

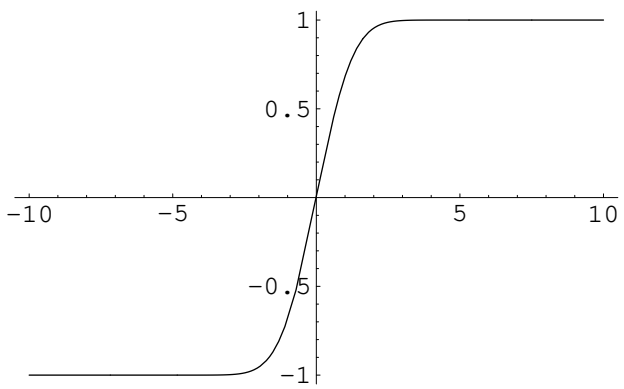
c. (4 points)

```
In[12]:= DSolve@8y'@xD == y@xD^2 - x * y@xD, y@0D == c<, y@xD, xD
```

Out[12]= 99y@xD fi − $\frac{2\,c\,a^{-\frac{x^2}{2}}}{-2 + c\,\sqrt{2\,p}\,\mathrm{ErfA}\,\frac{x}{\sqrt{2}}\,E}$ ==

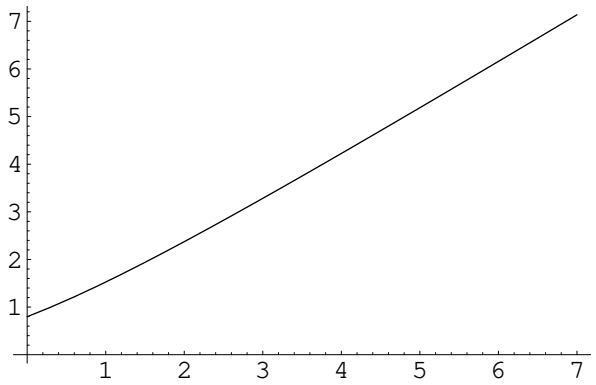
What the *&^&* is this Erf thing?

```
In[13]:= Plot@Erf@x  Sqrt@2DD, 8x, -10, 10<D;
```



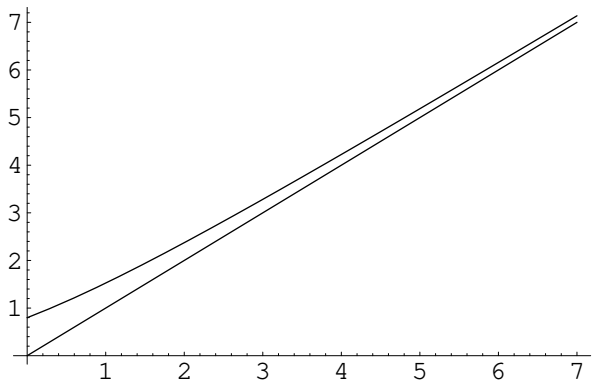
Doesn't really matter, I guess. The point is that ErfA$\frac{x}{\sqrt{2}}$E increases from − 1 to 1 as x goes from negative infinity to infinity. Therefore, if c = y@0D is less than $\sqrt{2\,p}$ then it's clear that y@xD is defined for all x H that is, the denominator never vanishes L and that y@xD converges to zero as x goes to infinity H because the numerator of y@xD definitely goes to zero while the denominator doesn't L If y@0D > $\sqrt{2\,p}$, then at some finite x the denominator vanishes and the solution goes to infinity along a vertical asymptote H this is different than what I predicted from the direction field L It's harder to tell just what happens if y@0D = $\sqrt{2\,p}$, so I plot that particular solution :

```
PlotA-        2  2  p E^(-x^2/2)
        - 2 +  2  p   2 p ErfA x/2 E , 8x, 0, 7<E;
```



This looks pretty close to a straight line, so I compare with the line y=x that looked important in the direction field:

```
PlotA9-        2  2  p E^(-x^2/2)
        - 2 +  2  p   2 p ErfA x/2 E , x=, 8x, 0, 7<E;
```



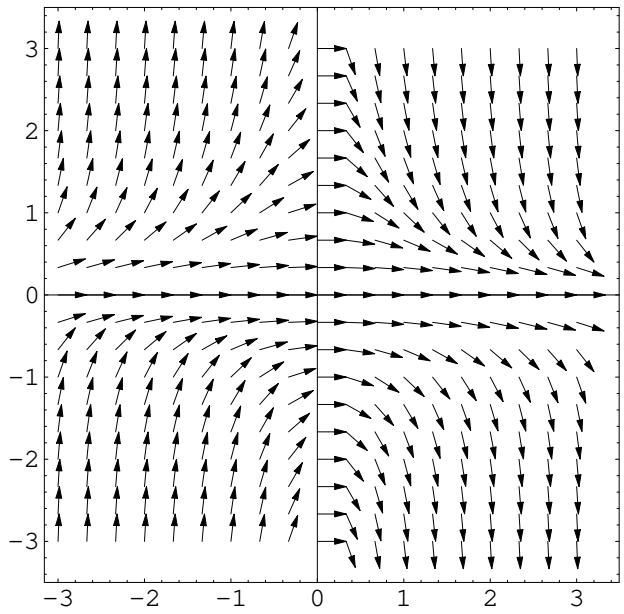
Sure enough, the "dividing" solution is asymptotic to the line y=x.

Problem 12.

a. (3 points)

```
Needs@"Graphics`PlotField`"D
```

```
PlotVectorField@81, - t * x^2<, 8t, - 3, 3<, 8x, - 3, 3<, Axes -> True,
    ScaleFunction fi H1 &L, Ticks fi None, Frame fi True, AspectRatio fi 1, PlotPoints -> 19D
```




```
Graphics
```

Looks like x = 0 is a constant solution.Moreover, it looks like solutions x[t] that start above x=0 will converge to zero as t goes to either positive or negative infinity,whereas solutions x[t] that start below x=0 will converge to negative infinity.

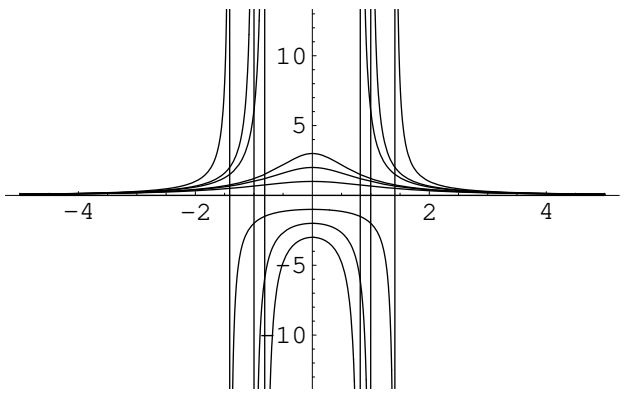
b. (3 points)

I'm going to get the solution to the ODE in a slightly different form than is asked for in the book.

```
soln = First@8x@tD . DSolve@8x'@tD == - t * x@tD^2, x@0D == c<, x@tD, tDD
```

$$\frac{2\,c}{2 + c\,t^2}$$

```
Plot@Evaluate@Table@soln, 8c, - 3, 3, 1<DD, 8t, - 5, 5<D;
```



So in fact, my predictions from the direction field weren't quite
right. It's true that any solution satisfying x[0] >0 will
converge to 0 as t goes to positive or negative infinity.
However, any negative solution appears to have to vertical asymptotes and
to go to negative infinity along both of them−−i.esolutions that start below
x−axisgo to negative infinity at some *finite* positive and negative
values of t.

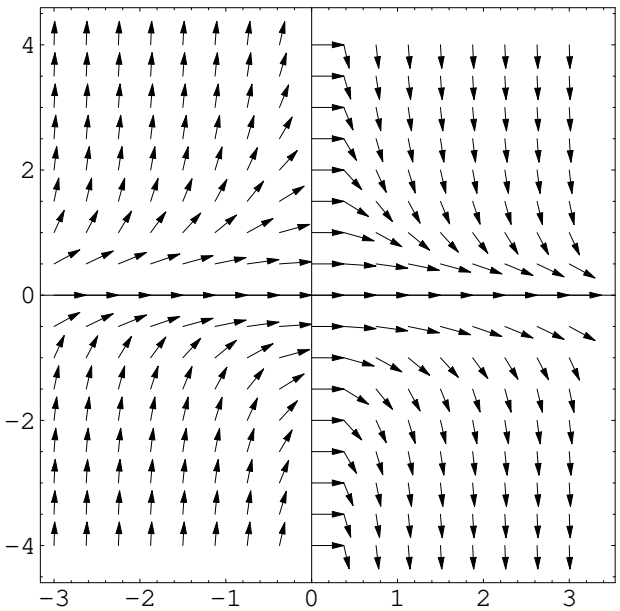
c. (4 points)

Five types of solution curves:
1. x[t] < 0, defined for all t in some finite interval symmetric about the
   y−axis,and x[t] tends to negative infinity at each end of the interval.
2. x[t] = 0 for all t
3. x[t] > 0, defined for all t and tending to zero as t goes to ±¥.
4. x[t] > 0 everywhere decreasing, and defined for t in an interval of the form [a,¥)
   where a > 0; asymptote at t = a.
5. x[t] > 0 everywhere, increasing and defined for t in an interval of the form (−¥,−a]
   where a > 0; asymptote at t = −a.

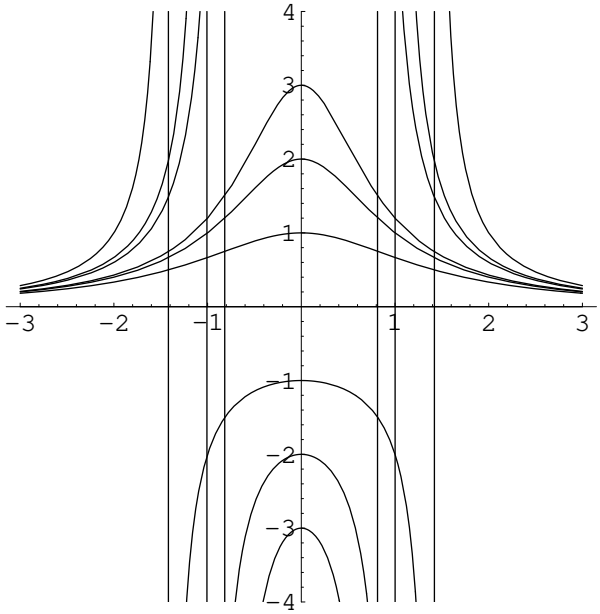
d. (2 points)

This 'Show' command is a little tricky−−tooksome playing around to get a good
picture.

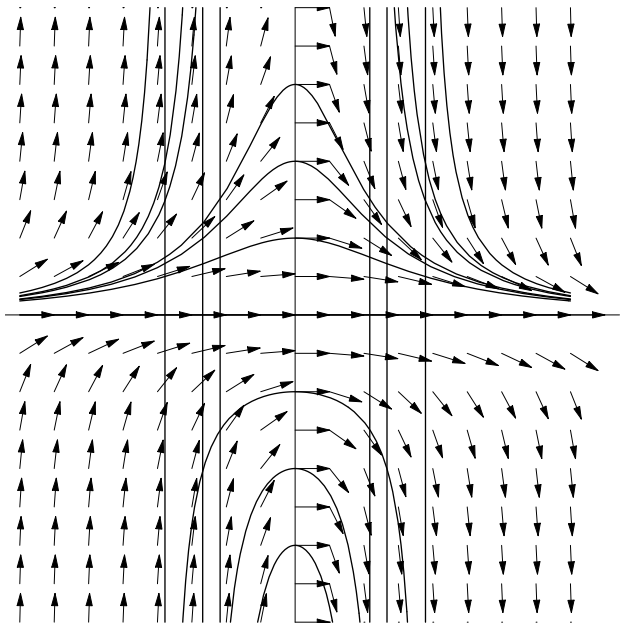
```
plot1 = PlotVectorField@81, - t * x^2<, 8t, - 3, 3<, 8x, - 4, 4<, PlotPoints -> 17,
    ScaleFunction fi H1 &L, Axes -> True, Ticks fi None, Frame fi True, AspectRatio fi 1D;
```



```
plot2 = Plot@Evaluate@Table@soln, 8c, - 3, 3, 1<DD, 8t, - 3, 3<,
  PlotRange -> 8- 4, 4<, AspectRatio -> 1D;
```




```
Show@plot1, plot2, PlotRange -> 8- 4, 4<, Frame -> False, Ticks -> NoneD;
```



Problem 16.

a. (3 points)

```
pop@t_, a_, b_, c_D = Expand@First@p@tD . DSolve@
                        8p'@tD == a p@tD + b, p@0D == c<,
                      p@tD, tDDD
```

$$-\frac{b}{a}+\frac{b^{a\,a\,t}}{a}+c^{a\,a\,t}$$

The parameter c is simply the population of Nevada in 1950––i.ethe initial value. The parameter a determines the contribution to population growth coming from births –deaths , and b is the contribution to growth coming from (people moving in) –(people moving out).

b (3 points)

```
Solve@8pop@0, a, b, cD == 285.3, pop@10, a, b, cD == 488.7,
            pop@20, a, b, cD == 800.5<, 8a, b, c<D
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
88c fi 285.3, b fi 4.11623, a fi 0.0427187<,
 8c fi 285.3, b fi 4.11623 - 60.5425  , a fi 0.0427187 - 0.628319  <,
 8c fi 285.3, b fi 4.11623 + 60.5425  , a fi 0.0427187 + 0.628319  <,
 8c fi 285.3, b fi 4.11623 - 121.085  , a fi 0.0427187 - 1.25664  <,
 8c fi 285.3, b fi 4.11623 + 121.085  , a fi 0.0427187 + 1.25664  <,
 8c fi 285.3, b fi 4.11623 - 181.628  , a fi 0.0427187 - 1.88496  <,
 8c fi 285.3, b fi 4.11623 + 181.628  , a fi 0.0427187 + 1.88496  <,
 8c fi 285.3, b fi 4.11623 - 242.17  , a fi 0.0427187 - 2.51327  <,
 8c fi 285.3, b fi 4.11623 + 242.17  , a fi 0.0427187 + 2.51327  <,
 8c fi 285.3, b fi 4.11623 + 302.713  , a fi 0.0427187 + 3.14159  <<
```

Only the first in this list of solutions gives real values for a,b, and c. I'll use that solution.

c. (4 points)

```
population@t_D = pop@t, 0.042718, 4.1162, 285.3D
```

$$-96.3575+381.658^{\,a^{0.042718\,t}}$$

Now Icheck the population in 1950 and 1990 predicted by this model:

```
population@10D
```

```
152.615
```

```
population@30D
```

```
1278.44
```

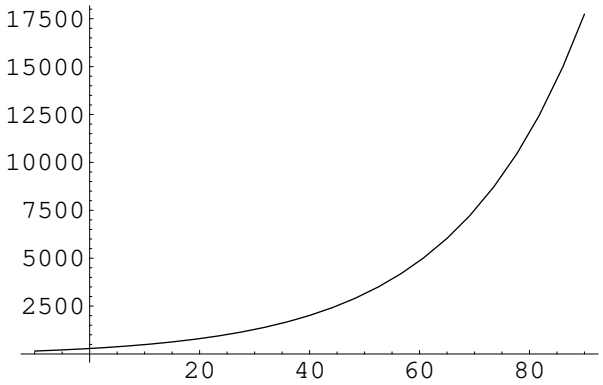
These are at least close to the true values. To predict when the population reaches three million, I'll guess that it'll be fifteen years after 1990 (since the population nearly tripled in the twenty years between 1970 and 1990).

```
FindRoot@population@tD == 3000, 8t, 45<D
```

```
8t fi 49.0065<
```

So I'm predicting a population of three million by 2009
Given the real data for 1990 and 1950 this model seems to be projecting too much population growth. I'd probably guess that it'll be a few years after 2009 when the population really does reach three million. Based on the formula above, it looks like I'd have to decrease both a and b to get a more accurate population model.

```
Plot@population@tD, 8t, - 10, 90<D;
```



If the model is right, Nevada will get pretty crowded in time!

Problem 17.

a. (2 points)

Mathematica is really overkill for this part...

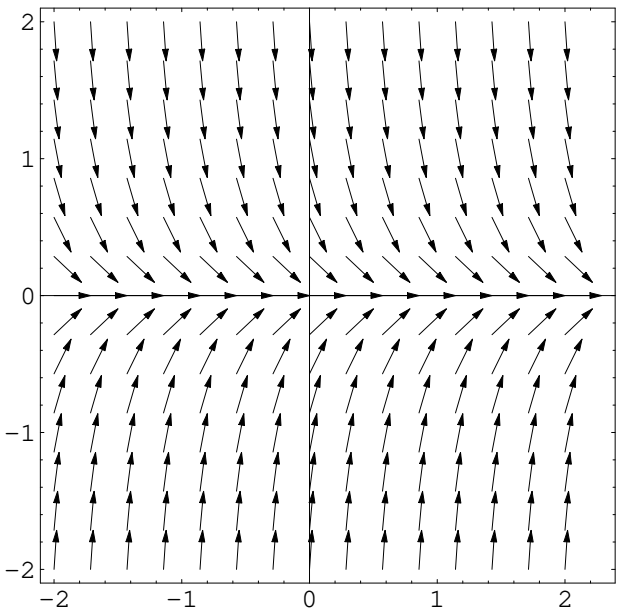
```
Solve@Ha - 1L x - x^3 == 0, xD
```

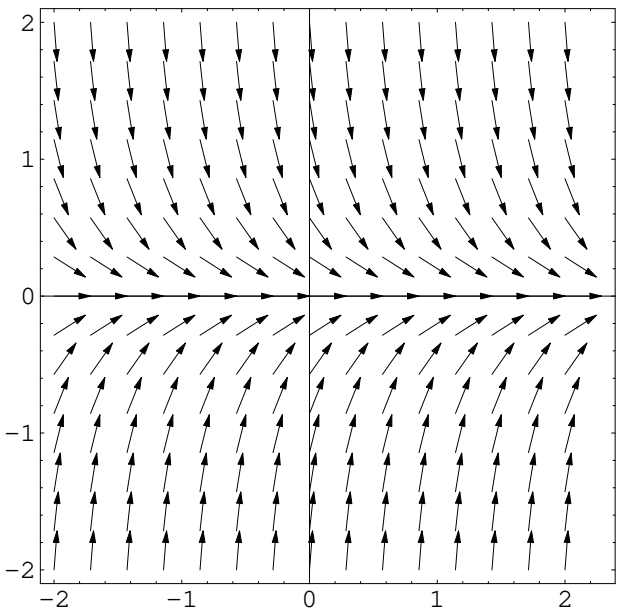
```
98x fi 0<, 9x fi - 1
2
!!!!!!!!!!!!!
a + 1, x fi 1
2
!!!!!!!!!!!!!
a + 1=
```

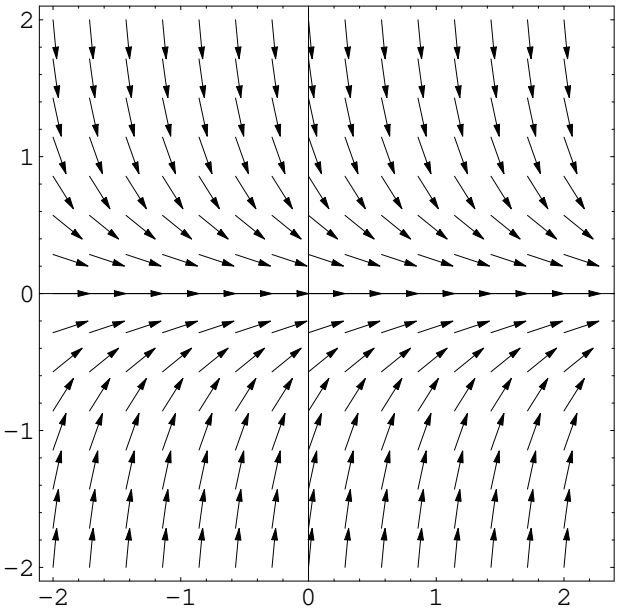
That is, we get three real roots when a>1, and only one real root when a<1. When a=1, we also get one real root, but with multiplicity three.

b. (2 points)

```
Table@PlotVectorField@
        81, Ha - 1L x - x^3<, 8t, - 2, 2<, 8x, - 2, 2<, Axes -> True,
    ScaleFunction fi H1 &L, Ticks fi None,
   Frame fi True, AspectRatio fi 1, PlotPoints -> 15D, 8a, - 2, 0<D;
```


2
1
0
−1
−2
−2
−1
0
1
2

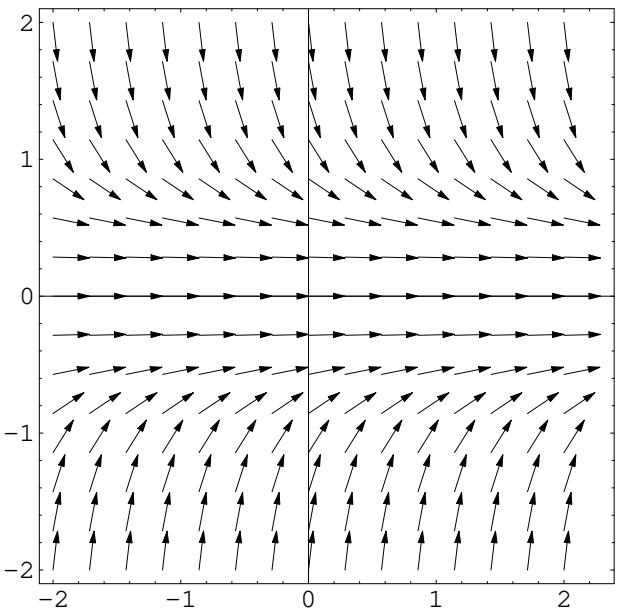

2
1
0
−1
−2
−2
−1
0
1
2



The picture is more or less the same in all three cases: x=0 is a stable equilibrium solution.

c. (2 points)

```
PlotVectorField@
        81, - x^3<, 8t, - 2, 2<, 8x, - 2, 2<, Axes -> True,
    ScaleFunction fi H1 &L, Ticks fi None, Frame fi True, AspectRatio fi 1, PlotPoints -> 15D
```

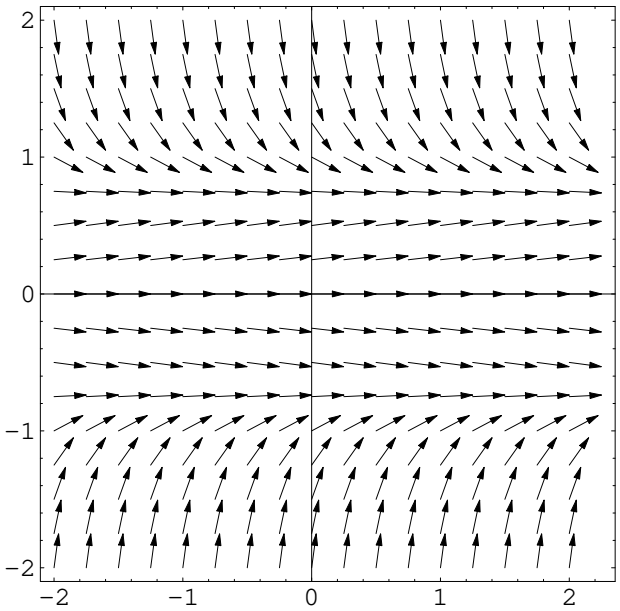



```
Graphics
```

x=0 is still a (barely) stable equilibrium.

d. (3 points)

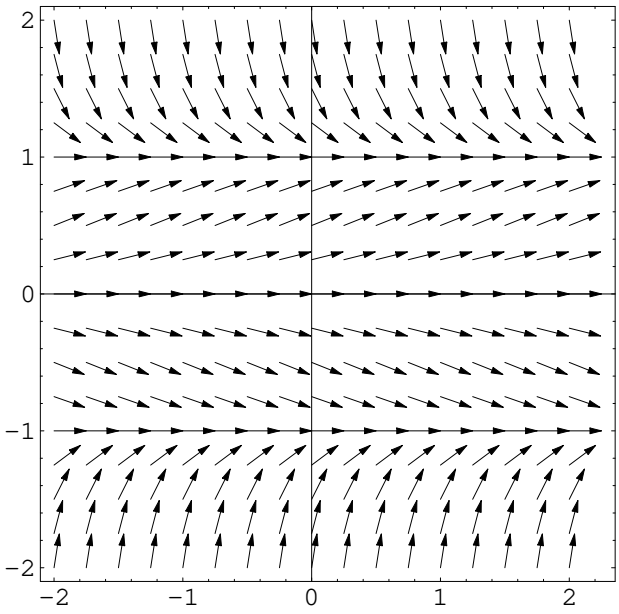
```
PlotVectorField@
        81, .5 x - x^3<, 8t, - 2, 2<, 8x, - 2, 2<, Axes -> True,
    ScaleFunction fi H1 &L, Ticks fi None, Frame fi True, AspectRatio fi 1, PlotPoints -> 17D
```




```
Graphics
```

Now x = 0 is an unstable equilibrium, but there are two stable equilibrium solutions at x = − .5

```
PlotVectorField@
        81, x - x^3<, 8t, - 2, 2<, 8x, - 2, 2<, Axes -> True,
    ScaleFunction fi H1 &L, Ticks fi None, Frame fi True, AspectRatio fi 1, PlotPoints -> 17D
```




```
Graphics
```

This is similar to the last case, except that the stable equilibria have migrated to x=±1.

e. (3 points)

When `a`<0, we had a single stable equilibrium solution at x=0. When `a` increased past 0, we suddenly had two stable equilibrium that started just above and below x=0 and moved further from x=0 as `a` increased.