Name: ________________________________ **Instructor:** Jeffrey Diller

Math 230: Ordinary Differential Equations
Spring Semester 2001
Final Exam
Tuesday, May 8

This Examination contains ten problems on nine sheets of paper including the front cover. Do all your work on the paper provided and show your computations. **When it makes sense to do so, please box your answers—this makes it much easier for me to see what you're doing.**

Scores

Question	Possible	Actual
1	10	
2	5	
3	20	
4	30	
5	10	
6	20	
7	15	
8	10	
9	20	
10	10	
Total	150	

GOOD LUCK

...and enjoy your summer!

1. Solve the initial value problem

$$y' + ty - t = 0, \qquad y(-1) = 2.$$

(10 points)

2. Consider *but do* **NOT** *try to solve* the initial value problem

$$(1-t)y'' + (2-t)y = \ln(16 - t^2), \qquad y(3) = 5, y'(3) = 10^6.$$

On what interval is this problem guaranteed to have a solution? (5 points)

3. Suppose that $y(t)$ solves the initial value problem

$$y' = \sqrt{t + y^2}, \qquad y(0) = 2.$$

(a) Use Euler's method with a single step to approximate $y(.1)$. (10 points)

(b) Estimate the difference between your approximation and the actual value of $y(.1)$—that is, estimate the local truncation error. You needn't simplify any of your arithmetic. (10 points)

4. Consider the inhomogeneous differential equation.

$$y'' + 2y' + 2y = \sin 2t + t$$

(a) Find the general solution of the corresponding *homogeneous* equation. (10 points)

(b) Find the form of the general solution of the (inhomogeneous) equation, and circle any coefficients that must be determined. *Do* **NOT** *try to determine the coefficients.* (10 points)

(c) Convert the inhomogeneous equation into an equivalent 2×2 system of first order ODE's, and write the system in matrix form. (10 points)

5. Find the general solution of the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(10 points)

6. Consider (but do **NOT** try to solve) the autonomous differential equation

$$y' = (y - 10)(y - 20)(y - 30)^2.$$

(a) Find and classify the critical points of this equation. (10 points)

(b) Given a number $0 \leq y_0 \leq 20$ and a solution $y(t)$ with initial value $y(0) = y_0$, what happens to the value of $y(t)$ as t increases? Note that the answer might depend on y_0. (10 points)

7. Suppose that A is a 2×2 constant matrix and that the direction field for $\mathbf{x}' = A\mathbf{x}$ is as follows:

(a) Highlight any visible eigenspaces in the direction field. (6 points)

(b) Write down a plausible general solution $\mathbf{x}(t)$ for this system. Of course your numbers won't be exactly right—close is good enough. (9 points)

8. Here is a phase portrait/direction field for a certain 2×2 (non-linear) autonomous system of ODE's.

Identify each critical point of the system along with its type and stability. (10 points)

9. Consider the non-linear autonomous system

$$\begin{aligned} x' &= (x-y)(y-2) \\ y' &= xy - 1 \end{aligned}$$

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(a) Find the critical points of this system. (10 points)

(b) For one of the critical points (your choice) write down the corresponding linear system and draw conclusions (if any) about the stability of that point. (10 points)

10. Having become expert at solving systems of ODE's, you amuse yourself by finding the general solution to a linear, homogeneous 2×2 system of ODE's specified in a random homework problem from chapter 7 in Boyce and DiPrima. You obtain

$$\mathbf{x}(t) = C_1 e^t \begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + C_1 e^t \begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

There is, of course, no chance that you have made a mistake. However, out of concern for your fellow, more fallible classmates, you compare your solution with the answer given in the back of the book to make sure that the book is right. The book says:

$$\mathbf{x}(t) = Ae^t \begin{pmatrix} \cos 3t \\ \cos 3t - \sin 3t \end{pmatrix} + Be^t \begin{pmatrix} \sin 3t \\ \cos 3t + \sin 3t \end{pmatrix}.$$

Is the book right? Why or why not? (10 points)