MATH 261 – LINEAR ALGEBRA

FALL 1999 (EXAM 2)

(1) Recall that P_n denotes the space of real polynomials of degree at most n and consider the linear transformation $T : P_n \to P_{n+1}$ given by $(Tf)(x) = \int_0^x f(t)dt$. Write down the matrix of T relative to the canonical bases of P_n and P_{n+1}.

(2) Let $T : V \to V$ be a linear operator, where V is finite-dimensional. A subspace W of V is said to be T – *invariant* if $TW \subset W$. Start with a basis for W and then complete it to a basis β of V. What does the matrix of T relative to β look like?

(3) Give bases for the kernel and the range of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T(x_1, x_2, x_3) = (3x_1 + x_2 - 4x_3, 2x_2 + x_3, 3x_1 - 2x_3, x_2 - x_3).$$

(4) Consider the subspaces W_1 and W_2 of $\mathbb{R}^{13}$, $\dim W_1 = 8$, $\dim W_2 = 7$. What are the possible dimensions of $W_1 \cap W_2$? Explain your answer carefully and give examples to illustrate some of the possibilities.

(5) a) Define what it means for two vector spaces to be isomorphic. b) Exhibit an isomorphism between $\mathbb{R}^{n+1}$ and the space of polynomials of degree at most n . c) Prove that if two finite-dimensional vector spaces are isomorphic then they have the same dimension. d) Does the converse of c) hold ? You must settle d) by either giving a proof or by exhibiting a counter-example.

(6) Write an essay about products of matrices and how they relate to composition of linear transformations. Your essay must include examples and the proof of at least one important theorem. You can borrow responsibly from known sources but the examples should be your own.