Math 261 – Linear Algebra
Fall 2000 (Exam 2)

1. *(20 points)* Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T(x_1, x_2, x_3) = (3x_1 + x_2 - 4x_3, 2x_2 + x_3, 3x_1 - 2x_3, x_2 - x_3).$$

(a) What is the matrix of T with respect to the standard ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^4$?

(b) Find bases for the kernel (null space) and the range of T.

2. *(10 points)* Let P_n denote the space of real polynomials of degree at most n. It has a standard ordered basis $\{1, x, \dots, x^n\}$. Consider the linear transformation $T : P_n \to P_{n+1}$ given by $(Tf)(x) = \int_0^x f(t)dt$. Write down the matrix of T relative to the standard ordered bases of P_n and P_{n+1}. What is the rank of T?

3. *(10 points)* Let $T : V \to V$ be a linear operator, where V is finite-dimensional. A subspace W of V is said to be T – *invariant* if $TW \subseteq W$. Start with a basis $\{w_1, \dots, w_p\}$ for W and then complete it to a basis $\beta = \{w_1, \dots, w_p, w_{p+1}, \dots, w_n\}$ of V. What does the matrix of T relative to the (ordered) basis β look like?

4. *(15 points)*

(a) Consider subspaces W_1 and W_2 of $\mathbb{R}^{13}$ with $\dim W_1 = 8$, $\dim W_2 = 7$. What are the possible dimensions of $W_1 \cap W_2$?

(b) Consider a linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^3$. What are the possible values of the nullity of T?

5. *(30 points)*

(a) Define what it means for two vector spaces to be isomorphic.

(b) Exhibit an isomorphism between $\mathbb{R}^{n+1}$ and the space of real polynomials of degree at most n.

(c) Suppose that $\theta: V \to W$ is an isomorphism between F-vector spaces V and W. Prove that if $\{v_1, \dots, v_n\}$ is a basis of V, then $\{\theta(v_1), \dots, \theta(v_n)\}$ is a basis for W.

6. *(15 points)* Let $S: V \to W$ and $T: W \to U$ be linear transformations between finite-dimensional F-vector spaces. Let α, β and γ denote ordered bases of V, W and U respectively. How is the matrix $[S]_\alpha^\beta$ defined? State and carefully prove the connection between $[S]_\alpha^\beta$, $[T]_\beta^\gamma$ and $[TS]_\alpha^\gamma$.

Bonus Question *(10 points)* Let V be a finite dimensional vector space over a field F. Let $\Psi: V \to V^{**}$ be the natural isomorphism of V with its double dual space defined by $\Psi(v) = \hat{v}$ for $v \in V$ where $\hat{v}(f) = f(v)$ for $f \in V^*$. Let $T: V \to V^*$ be a linear transformation and $T^t: V^{**} \to V^*$ be its transpose, defined by $T^t(f) = fT$ for $f \in V^{**}$. Determine whether $T^t\Psi = T$ as linear transformations $V \to V^*$. If so, prove it. If not, find necessary and sufficient conditions on the matrix $A = [T]_\beta^{\beta^*}$ of T (with respect to an ordered basis β of V and corresponding dual basis β^* of V^*) for $T^t\Psi = T$ to hold.