Math 261 – Linear Algebra

Fall 2000 (Final Exam)

1. *(30 points)* Consider the linear transformation $T\colon \mathbf{R^4} \to \mathbf{R^3}$ given by

$$T(x_1, x_2, x_3, x_4) = (3x_1 - 2x_2 + 4x_3 - x_4, 2x_1 + 3x_2 - 5x_3 + 4x_4, 4x_1 - 7x_2 + 13x_3 - 6x_4).$$

(a) Determine a basis for the null space of T.

(b) Determine a basis for the range of T.

2. *(30 points)* Explicitly find an ordered basis β for $\mathbf{R^2}$ such that the matrix $[T]_\beta^\beta$ of the linear transformation $T : \mathbf{R^2} \to \mathbf{R^2}$ given by $T(x, y) = (x+2y, 2x+y)$ is a diagonal matrix.

3. *(30 points)* Let V be a finite-dimensional vector space.

(a) Explain what is meant by saying that linear maps $S, T : V \to V$ are similar.

(b) Prove that similar linear maps $S, T : V \to V$ have the same rank.

4. *(30 points)*

(a) Let V, W be finite-dimensional vector spaces over a field F. Prove that there is a *surjective* linear map $T\colon V \to W$ if and only if $\dim(V) \geq \dim(W)$.

(b) Is there an *injective* linear transformation from the space of all $n \times n$ symmetric real matrices into $\mathbf{R^{2n}}$?

5. *(30 points)*

(a) Define the determinant of an $n \times n$ matrix over a field F.

(b) Let $A^{(n)}$ be the $n \times n$ real matrix with 2's on the main diagonal, 1's on the diagonals above and below the main diagonal and zero's everywhere else i.e with entries $(A^{(n)})_{i,i} = 2$, $(A^{(n)})_{i,j} = 1$ if $j - i = \pm 1$ and $(A^{(n)})_{i,j} = 0$ otherwise. Prove that $\det(A^{(n)}) = n + 1$. (Hint: Let $a_n := \det(A^{(n)})$. Show by expansion along the first row that $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 3$).