Math 261 December 16, 2002

Final

The total number of points possible on this exam is 175. 165 will be considered a perfect score.

1. (15 points)

   (a) Show that the system

   $$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 2 \\ 3x & + & y & + & 3z & = & 2 \\ 7x & - & y & + & az & = & -2 \end{array}$$

   has a unique solution if $a \neq 2$.

   (b) Find all solutions when $a = 2$.

   (c) List at least three ways you could solve the system if $a = 2002$.

2. (23 points) Let T be the linear transformation from $\mathbf{R}^6$ to $\mathbf{R}^4$ given by $TX = AX$ where

   $$A = \begin{bmatrix} 1 & 1 & 0 & -7 & 0 & -1 \\ 0 & 1 & 1 & 23 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & -1 & -7 \end{bmatrix}$$

   and $X = (x_1, \ldots, x_6)^t$.

   (a) Find a basis for the null space of T.

   (b) Find a basis for the range of T.

   (c) Find the null space of T^t.

   (d) Find a basis for the range of T^t.

   (e) Find the annihilator of the range of T.

   (f) What are the rank and nullity of T?

   (g) What are the rank and nullity of T^t?

3. (12 points) Let V_n be the space of real polynomials of degree at most n. Let k be a fixed positive integer, $k < n$. For each of the following subsets of V_n, determine whether it is a subspace and if so compute its dimension and find a basis for it.

   (a) $S = \{p \in V_n : p \text{ has degree } k\}$.

   (b) $S = \{p \in V_n : p \text{ has degree } \leq k\}$.

(c) $S = \{p \in V_n : p(x) \leq 0, \text{ all } x\}$.

(d) $S = \{p \in V_n : p(0) + 3p''(0) = 0\}$.

4. (10 points) If V and W are finite dimensional vector spaces and $T : V \to W$ is an isomorphism, prove that T maps a basis of V onto a basis of W.

5. (10 points) Show that the matrices $\begin{bmatrix} 1 & 0 \\ 5 & 11 \end{bmatrix}$ and $\begin{bmatrix} 1 & 11 \\ 0 & 5 \end{bmatrix}$ are not similar.

6. (10 points) Use properties of the determinant to evaluate $\begin{bmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{bmatrix}$.

7. (10 points) Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Find $\operatorname{adj} A$.