Math 261

Study Guide for Final

Due at 8:00 a.m., December 16, 2001

Create a study guide for the course. It should include key definitions and theorems and a brief outline of important techniques. It should also indicate relations among different parts of the course (for example, how solutions of systems of equations are related to subspaces and dimensions). You may include other things you consider important for understanding the course. Your study guide should be organized in a reasonable way. This will be treated as a 50 point homework assignment. In particular, you are free to discuss it with anyone, but you must write it yourself.

Take-home problems for the Final

Due at 8:00 a.m., December 16, 2001

You may consult your course notes, homework, Hoffman and Kunze and ATLAST. You may use Maple, Matlab or Mathematica as an experimental tool on any part of the problem. If you do, please indicate how on your exam. You may not consult any other books or notes. You may not discuss the exam with anyone except me.

Be sure to explain your answers. For example, if you are asked to find a basis for a certain vector space, I expect you not only to find it but also to prove that it is a basis. Make sure that your explanations are clear. The word *show* is a synonym for *prove*.

1. Let A be an $m \times n$ matrix over a field F. Suppose $\operatorname{rank}(A) = r$ and the system of equations $AX = Y$ has a solution X_0. Describe all solutions to the system. (*Hint*: Suppose X_1 is another solution. What can you say about $X_0 - X_1$?)

2. (a) Let $A, B \in F^{n\times n}$. Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. (The trace tr of a matrix is defined in Example 19 on page 98 of Hoffman and Kunze.)

   (b) Suppose A and B are similar. Show that $\operatorname{tr}(A) = \operatorname{tr}(B)$.

   (c) Let V be a finite dimensional vector space, $T : V \to V$ a linear operator. Let $\mathcal{B}, \mathcal{B}'$ be ordered bases of V. Show that $\operatorname{tr}[T]_{\mathcal{B}} = \operatorname{tr}[T]_{\mathcal{B}'}$. Use this to define the **trace** of T, $\operatorname{tr} T$.

   (d) Let V be the space of all 2×2 matrices over a field F and let $P \in V$ be fixed. Define a linear operator T on V by $T(A) = PA$. Show that $\operatorname{tr} T = 2 \operatorname{tr} P$.

3. Let V be the space of all real polynomials and let f be the linear functional on V defined by $f(p) = \int_0^1 p(x)\,dx$. Let $D : V \to V$ be differentiation. What is $D^t f$?

4. (a) Prove: If A is an $n \times n$ matrix over $\mathbf{C}$ there are at most n distinct scalars c such that $\det(cI - A) = 0$.

   (b) Give an example where there are n distinct scalars as in (a).

   (c) Show that if $\det(cI - A) = 0$ there is an $x \in \mathbf{C}^n$, $x \neq 0$ such that $Ax = cx$.

5. An $n \times n$ matrix over $\mathbf{R}$ is *orthogonal* if $A^t A = I$.

   (a) Show that if A is an $n \times n$ orthogonal matrix over $\mathbf{R}$ then $\det A = \pm 1$.

   (b) Give an example of a 2×2 orthogonal matrix A over $\mathbf{R}$ such that $\det A = -1$.

6. (a) Show that if the entries of A are integers and $\det A = \pm 1$ then the entries of A^{-1} are integers.

   (b) Suppose the entries of A are integers and A is nonsingular. What can you say about the entries of A^{-1}?

   (c) Show that if the entries of A and A^{-1} are integers then $\det A = \pm 1$.