Mathematics 262
Solutions to practice problems

Assignment:

- Section 8.4: 12
- Section 8.5: 1, 3, 4, 10, 12

§8.4, #12: It's probably worthwhile thinking about how to do this problem without the material from Section 8.5. Using that material, though, makes things easier. The problem asks you to show that if V is a finite-dimensional inner product space, then for every linear operator on T which is both self-adjoint and unitary, there is a subspace W of V so that T acts as the identity I on W and T acts as $-I$ on the orthogonal complement $W^{\perp}$. In other words, I want to find a basis for V with respect to which the matrix for T is of the form

$$A = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & -1 \end{bmatrix}.$$

If T is both self-adjoint and unitary, then the main theorems of Section 8.5 apply: there is a basis for V consisting of eigenvectors for T. What are the eigenvalues for V? Since T is self-adjoint, they're real. Since T is unitary, they have norm 1. Hence 1 and -1 are the only eigenvalues. So I take my basis of eigenvectors and order it by putting the eigenvectors corresponding to 1 first, and then the eigenvectors corresponding to -1. This does the trick.

§8.5, #1: All you have to do is find an orthonormal set of eigenvectors and make them the columns of the matrix P. (It wouldn't hurt to check that P^tAP is actually diagonal, too.) Here are the matrices P for the three matrices in the problem:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}.$$

For the third one, you might be able to recognize the matrix A as a reflection, and in particular, reflection across the line through the origin at angle $\frac{\theta}{2}$ from the horizontal. So any vector parallel to this line is an eigenvector with eigenvalue 1; any vector perpendicular to this is an eigenvector with eigenvalue -1. If you don't see this, you have to use a few trig identities to get the right answer.

§8.5, #3: The entries of D are the eigenvalues of A. The characteristic polynomial is $x^3 - 9x^2 - 6x$. This factors as $x(x^2 - 9x - 6)$, and so has roots 0,

$\frac{9\pm\sqrt{105}}{2}$. There are several choices for D, depending on the order in which we put the eigenvalues; one choice would be

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{9+\sqrt{105}}{2} & 0 \\ 0 & 0 & \frac{9-\sqrt{105}}{2} \end{bmatrix}.$$

§8.5, #4: To show that T is normal, you have to check that $AA^* = A^*A$. Then you find the eigenvalues for A, which are $1+i$ and $1-i$. Then you find corresponding eigenvectors, which are $(1,1)$ and $(-1,1)$. Then you normalize to get an orthonormal basis:

$$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

§8.5, #10: We did this in class using the spectral theorem. A is a *positive* matrix if A is Hermitian and $X^*AX > 0$ for every nonzero column vector X. Since A is Hermitian, then V has an orthonormal basis consisting of eigenvectors for A; with respect to that basis, A becomes a diagonal matrix D. Since X^*AX is always positive, then $\varepsilon_i^* D\varepsilon_i$ is positive for each i, where ε_i is the ith standard basis vector. On the other hand $\varepsilon_i^* D\varepsilon_i$ is the ith entry on the diagonal of D, so we can conclude that all of the entries of D are positive. Therefore D has a square root. Changing back to the original basis gives a square root for A. (I've left out lots of details here, and you might want to fill them in.)

§8.5, #12: Let T be normal. If I view T as operating on a complex vector space V, then by the main theorem (Theorem 22 in the book), there is an orthonormal basis of V consisting of eigenvectors of T. Note the word "orthonormal."

(This isn't quite enough. I should really start with a normal operator T acting on any vector space V. If V happens to be complex, the previous paragraph solves the problem. Otherwise, if V is real and if α and β are two eigenvectors corresponding to distinct eigenvalues c and d, then I have to extend the action of T on V to an action of T on a complex vector space W, note that α and β are still eigenvectors when viewed as elements of W, and then apply the argument of the previous paragraph.)