Mathematics 262
Homework 2 solutions

Assignment: Section 5.3, problems 1-3; Section 5.4, problem 3.

§5.3, #1: This is a computation. One method is to use expansion by minors along, say, the first column. You get

$$\begin{aligned}\det A &= 0 \cdot \det \left(\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right) + (-1)(-a) \cdot \det \left(\begin{smallmatrix} a & b \\ -c & 0 \end{smallmatrix}\right) + (-b) \cdot \det \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \\ &= abc - abc = 0.\end{aligned}$$

Note: you might have been tempted to add a multiple of the second row to the third to get a zero in the bottom right corner of the matrix (i.e., multiply the second row by $-\frac{b}{a}$ and add it to the third row). This isn't a valid argument, though, because you can't assume that you can divide by the element a in the ring K.

§5.3, #2: This is another computation. Here you can do the partial row reduction thing to cancel off the 1's in the first column. Subtracting the first row from the second and third and then expanding by minors along the first column gives

$$\begin{aligned}\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} &= \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{pmatrix} \\ &= (b-a)(c^2-a^2) - (b^2-a^2)(c-a) \\ &= (b-a)(c-a)(c+a) - (b-a)(c-a)(b+a) \\ &= (b-a)(c-a)\left((c+a)-(b+a)\right) \\ &= (b-a)(c-a)(c-b).\end{aligned}$$

§5.3, #3: The six permutations of degree 3, and their signs, are:

permutation	sign	permutation	sign
$(1,2,3)$	$+1$	$(1,3,2)$	-1
$(2,3,1)$	$+1$	$(2,1,3)$	-1
$(3,1,2)$	$+1$	$(3,2,1)$	-1

To figure out each sign, I just figured out how many transpositions were required to turn the permutation into $(1,2,3)$; for instance, $(3,1,2)$ takes two steps:

$$(3,1,2) \mapsto (3,2,1) \mapsto (1,2,3).$$

The formula for determinant is then:

$$\begin{aligned}&\det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \\ &= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} + A_{12}A_{23}A_{31} - A_{12}A_{21}A_{33} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}.\end{aligned}$$

§5.4, #3: For any $n \times n$ matrix A, $\det A = \det A^t$. If A is skew-symmetric, then $A^t = -A$, so we have $\det A = \det(-A)$. One can obtain $-A$ from A by multiplying each row of A by -1; using the fact that the determinant is n-linear, this gives the equation

$$\det A = \det(-A) = (-1)^n \det A.$$

Since n is odd, then we have $\det A = -\det A$, or equivalently, $2 \det A = 0$. Since we are working with complex numbers, we can cancel the 2, to conclude that $\det A = 0$. (Whenever a product is zero, one of the two factors must be zero. If A were defined over the field with two elements instead of the complex numbers, then $1 + 1 = 2$ would equal 0, so we would not be able to conclude that $\det A = 0$.)