Mathematics 262
"Homework 4" solutions

Assignment: Section 6.2, problems 1, 4, 6, 7

§6.2, #1: For the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

the characteristic polynomial is $(x-1)x$, so the characteristic values are 1 and 0. (This is true over both $\mathbf{R}$ and $\mathbf{C}$.) The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for the eigenvalue 1, and the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector for 2.

For the matrix

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix},$$

the characteristic polynomial is $x^2 - 3x + 5$, which has no roots over $\mathbf{R}$, so T has no characteristic values. The operator U, defined over $\mathbf{C}$, has characteristic values $\frac{3 \pm i\sqrt{11}}{2}$. The corresponding characteristic vectors are messy. Whenever you have a 2×2 matrix A with characteristic value c, then the rows of $cI - A$ are linearly dependent; since there are only two rows, this means that either one row is zero, or each row is a scalar multiple of the other. In general, this is a good way to check your work; if the algebra is a bit complicated, as in this case, you can also just look at, say, the second row. For the characteristic value $c = \frac{3+i\sqrt{11}}{2}$, $cI - A$ is

$$\begin{bmatrix} \frac{3+i\sqrt{11}}{2} - 2 & -3 \\ 1 & \frac{3+i\sqrt{11}}{2} - 1 \end{bmatrix}.$$

From the second row, I can see that $(\frac{3+i\sqrt{11}}{2} - 1, -1)$ is a characteristic vector. I could instead have used the first row to get the characteristic vector $(3, \frac{3+i\sqrt{11}}{2} - 2)$, which *is* a scalar multiple of the first vector, even though it may not look like it.

Similarly, the other characteristic value has associated characteristic vector $(\frac{3-i\sqrt{11}}{2} - 1, -1)$.

The characteristic polynomial of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is $x(x-2)$, so the roots are 0 and 2. 0 has eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and 2 has eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (Same for T and U.)

§6.2, #4: First I'll find the characteristic values, then for each of those, I'll find characteristic vectors. The characteristic polynomial is $f(x) = x^3 - x^2 - 5x - 3$. By inspection (i.e., guessing), I find that -1 is a root, so I factor out $x + 1$ and see what's left; I get that $f(x) = (x + 1)(x^2 - 2x - 3) = (x + 1)^2(x - 3)$. For -1, we need to find 2 linearly independent characteristic vectors; we start by writing down the matrix $-I - A$:

$$\begin{bmatrix} 8 & -4 & -4 \\ 8 & -4 & -4 \\ 16 & -8 & -8 \end{bmatrix}.$$

This obviously has rank 1, and hence nullity 2. The nullspace is spanned by the vectors $(1, 2, 0)$ and $(1, 0, 2)$ (for example). A characteristic vector for 3 is $(1, 1, 2)$. These three vectors together form a basis for $\mathbf{R}^3$, with respect to which T is diagonal. In fact, the matrix for T in this basis is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

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§6.2, #6: The given matrix has characteristic polynomial x^4, which has only one root, 0. So for this to be diagonalizable, the eigenspace for 0 must have dimension 4. Equivalently, the nullity of the matrix $0I - A = -A$ must be 4. Then $-A$ must be the zero matrix, so we must have $a = b = c = 0$.

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§6.2, #7: If T has n different eigenvalues $c_1, \ldots, c_n$, then since the characteristic polynomial for T has degree n, it must be $(x - c_1) \ldots (x - c_n)$. To apply our theorem, we only have to know that for each j, the eigenspace associated to c_j has the correct dimension, which in this case is 1. Eigenspaces always have dimension at least 1, and their dimension is at most the multiplicity of their eigenvalue as a root of the characteristic polynomial. In this case, that means that each eigenspace has dimension at most 1 (since there are no repeated roots of the characteristic polynomial), and hence has dimension exactly 1.

By the way, this is probably the easiest way to check that a matrix is diagonalizable, since in this case you just have to check that there are n distinct roots of the characteristic polynomial. When there are repeated roots, then you have to work harder to see whether it's diagonalizable.