Mathematics 262
Homework 5 solutions

Assignment: Section 6.2, problems 11, 12, 13; Section 6.3, problems 1, 5, 6, 8.

§6.2, #11: If N is not the zero matrix, then there is a (nonzero) vector α so that $N\alpha \neq \vec{0}$. Let $\beta = N\alpha$; then

$$N\beta = N(N\alpha) = N^2\alpha = \vec{0} = 0\beta.$$

This means that 0 is an eigenvalue, with eigenvector β. Since β is in the nullspace of N and α isn't, then α and β are linearly independent, and hence form a basis. With respect to this basis, the matrix for N looks like $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

§6.2, #12: [partial solution] Since we are working over the complex numbers, then the characteristic polynomial for A factors into linear terms, say as $(x-a)(x-b)$. If a and b are distinct, then we know (by problem 7, for instance, which was one of the practice problems last week) that A is diagonalizable, so A is similar to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

If $a = b$, then the eigenspace for a has dimension 1 or 2. If it has dimension 2, then by the theorem from class, A is diagonalizable, and hence is similar to $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (i.e., to $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a = b$). If the eigenspace for a has dimension 1, then A is not diagonalizable, so we'd better show that it's similar to $\begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}$. You should keep working on this, and turn it in next week.

§6.2, #13: Suppose that for some function f, we have $T(f) = cf$. Then

$$cf(x) = \int_0^x f(t)\ dt$$

for every x. (In particular, $f(0) = 0$.) Differentiating both sides and using the fundamental theorem of calculus gives

$$cf'(x) = f(x).$$

If $c = 0$, then obviously $f(x) = 0$ for all x. Otherwise, this is a separable differential equation; its only solution satisfying the initial condition $f(0) = 0$ is the zero function $f(x) = 0$. Since the zero function (i.e., vector) can never be an eigenvector, there are no eigenvectors, so there are no eigenvalues.

§6.3, #1: The minimal polynomial for I is $p(x) = x - 1$, since for a matrix A this becomes $p(A) = A - I$. If $A = I$, then $I - I$ is certainly 0. The minimal polynomial for the zero operator is $p(x) = x$.

§6.3, #5: Since T satisfies $g(x) = x^k$, then $p(x)$ must divide evenly into x^k. Therefore $p(x) = x^j$ for some j. Since the degree of $p(x)$ is at most n, then $j \leq n$. Well, if $T^j = 0$ for some j with $j \leq n$, then also $T^n = 0$.

§6.3, #6: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

§6.3, #8: Given a vector α, write it in coordinates as $\alpha = (\alpha_1, \alpha_2)$. Then P is defined by $P\alpha = \alpha_1$. It's easy to check that $P(\alpha + \beta) = \alpha_1 + \beta_1 = P(\alpha) + P(\beta)$, and that $P(c\alpha) = c\alpha_1 = cP(\alpha)$. (For one thing, you can write P as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.) The minimal polynomial is $p(x) = x^2 - x$.