Mathematics 262
Homework 6 solutions

Assignment: Section 6.2, problem 12 (cont'd); Section 6.4, problems 1, 4, 5, 9

§6.2, #12: When last we left our heroes, they had solved this problem except for the case when the matrix A has one eigenvalue a, with a 1-dimensional eigenspace. The characteristic polynomial for A is $(x-a)^2$; by the Cayley-Hamilton theorem, we have $(A-aI)^2 = 0$. Now we can apply problem 11 to the matrix $A-aI$, to conclude that $A-aI$ is similar to either the zero matrix or to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. In other words, for some invertible matrix P, we have

$$P(A-aI)P^{-1} = PAP^{-1} - P(aI)P^{-1} = PAP^{-1} - aI = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Hence PAP^{-1} is equal to either aI or to $\begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}$. The first case can't actually happen, because then the eigenspace for a would be 2-dimensional; therefore A must be similar to $\begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix}$.

§6.4, #1: (a) If T has an invariant subspace other than 0 and $\mathbf{R}^2$, then it must be one-dimensional, which means that there must be a nonzero vector α so that $T\alpha = c\alpha$ for some c. In other words, T must have an eigenvalue. But the characteristic polynomial for T is $x^2 - 3x + 4$, which has no real roots.

(b) The characteristic polynomial for A has complex roots $\frac{3 \pm \sqrt{9-16}}{2}$. These two distinct roots are the eigenvalues, and their one-dimensional eigenspaces are both invariant subspaces. (In fact, they are the only invariant one-dimensional subspaces, but you weren't asked to show that.)

§6.4, #4: By the theorem from class, if we want to know whether A is similar to a triangular matrix, we should try to find the minimal polynomial for A. To do this, first we find the characteristic polynomial; it ends up being $f(x) = x^3$. Since the minimal polynomial divides evenly into this, it must be $p(x) = x^r$, where r is 1, 2, or 3. In any case, it factors into linear terms, and so A is similar to a triangular matrix (with 0's on the diagonal—you always get the eigenvalues on the diagonal).

To find that triangular matrix, I guess we should look for a basis, as in the proof of the theorem. A has only one eigenvalue, 0, so the first basis element should be an eigenvector for 0. The eigenspace is 1-dimensional, spanned by $(1, 0, -1)$, so we'll let $\alpha_1 = (1, 0, -1)$. Now we look for a vector α_2 so that $(A - cI)\alpha_2$ is in the vector space spanned by α_1, where c is "some eigenvector of A." A only has one eigenvector, 0, so we can rewrite this equation as

$$A\alpha_2 = b\alpha_1.$$

If you solve this equation for α_2 (and in fact you might as well let $b = 1$, right?), you find that there are several possibilities for α_2, so you pick one: let $\alpha_2 = (1, 1, 0)$. (Another possibility is $(0, 1, 1)$, which is $\alpha_2 - \alpha_1$—any linear combination of α_1 and α_2 would have worked.) Now we want a vector α_3, not in the span of (α_1, α_2), which satisfies

$$A\alpha_3 = a\alpha_1 + b\alpha_2,$$

for some scalars a and b. Actually, at this point you know you're supposed to have a basis, so you could try anything for α_3 which is linearly independent from the other two, and see what happens. For instance, the vector $\alpha_3 = (1, 0, 0)$ is a solution to

$$A\alpha_3 = -2\alpha_1 + 2\alpha_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

With respect to the basis $(\alpha_1, \alpha_2, \alpha_3)$, the matrix A is

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(Note: the matrix A has rank 2, and so does this matrix. So this answer may in fact be correct. There are also lots of possible correct answers, unfortunately for the grader.)

§6.4, #5: If the matrix A satisfies $A^2 = A$, then it satisfies the polynomial $x^2 - x = x(x-1)$. Therefore the minimal polynomial for A divides evenly into $x(x-1)$, and so factors into distinct linear terms. Therefore A is similar to a diagonal matrix.

§6.4, #9: Yes, the space of polynomials is invariant under T: the integral of any polynomial is again a polynomial.

Yes, the space of differentiable functions is invariant under T: by the fundamental theorem of calculus, Tf is differentiable for every f.

No, the space of functions which vanish at $x = \frac{1}{2}$ is not invariant; for instance, if I let $f(x) = 2x - 1$, then $f(x)$ is in this subspace. On the other hand, $(Tf)(x) = x^2 - x$, which is not.

(On the other hand, the space of functions which vanish at 0 is invariant.)