Mathematics 262
Homework 7 solutions

Assignment:

- Prove the Cayley-Hamilton theorem for upper triangular matrices: if A is upper triangular with characteristic polynomial $f(x)$, then $f(A) = 0$.
- Section 6.5: 1(b), 2, 4 (extra credit)

Cayley-Hamilton: In order to show that $f(A) = 0$, I will use induction on n. Every 1×1 matrix B satisfies its characteristic polynomial (i.e., $f(B) = 0$), so that starts the induction. Now assume that every $(n-1) \times (n-1)$ upper triangular matrix satisfies its characteristic polynomial. Let A be an $n \times n$ upper triangular matrix with characteristic polynomial $f(x)$. One way to show that $f(A) = 0$ is to show that $f(A)\alpha = \vec{0}$ for every n-dimensional vector α. Even better, it's good enough to show that $f(A)\alpha = \vec{0}$ for every α in some basis. I'll use the standard basis: the basis with respect to which A is upper triangular. Call the basis elements $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$.

I can also write down a formula for $f(x)$: if the diagonal entries of A are $c_1, c_2, \ldots, c_n$, (these are not necessarily distinct) then

$$f(x) = (x - c_1)(x - c_2) \ldots (x - c_n).$$

Because A is upper triangular, the subspace W spanned by $\varepsilon_1, \ldots, \varepsilon_{n-1}$ is invariant under A; therefore A determines a linear operator $B : W \to W$. The matrix for B is A with the last row and column deleted; in particular, B is upper triangular. The characteristic polynomial for B is $g(x) = (x - c_1) \ldots (x - c_{n-1})$, and by induction, we know that $g(B) = 0$. In particular, $g(B)\varepsilon_i = \vec{0}$ for all $i \leq n - 1$.

Because of this, $g(A)\varepsilon_i = \vec{0}$ for all $i \leq n - 1$; since $f(A) = (A - c_n I)g(A)$, this means that $f(A)\varepsilon_i = \vec{0}$ for all $i \geq n - 1$. If we can show that $f(A)\varepsilon_n = \vec{0}$, we will be finished. Because A is upper triangular with $A_{nn} = c_n$, then $(A - c_n I)\varepsilon_n$ is a linear combination of $\varepsilon_1, \ldots, \varepsilon_{n-1}$. Therefore

$$\begin{aligned} f(A)\varepsilon_n &= g(A)(A - c_n I)\varepsilon_n \\ &= g(A)(r_1\varepsilon_1 + \cdots + r_{n-1}\varepsilon_{n-1}) \\ &= r_1 g(A)\varepsilon_1 + \cdots + r_{n-1} g(A)\varepsilon_{n-1} \\ &= \vec{0}. \end{aligned}$$

§6.5, #1(b): To find the invertible matrix P, I have to find a basis which consists of eigenvectors for both A and B. A has eigenvalues 0 and 2, with corresponding eigenvectors $(1, -1)$ and $(1, 1)$. You can easily check that these are also eigenvectors for B, with eigenvalues $1 - a$ and $1 + a$, respectively. Now let $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. One can check that $P^{-1}AP$ and $P^{-1}BP$ are diagonal (and, of course, the diagonal entries are the eigenvalues of the matrices). Since eigenvalues are not unique, and since they could have been chosen in either order, there are many correct answers here.

§6.5, #2: Since the matrices are defined over the complex numbers, they are all similar to triangular matrices. Since they commute, they are simultaneously triangulable. So the question becomes, how many linearly independent upper triangular matrices are there? Since the matrices are 3×3 and upper triangular, there are 6 possible nonzero entries; therefore, there are at most 6 linearly independent matrices in $\mathcal{F}$. In the $n \times n$ case, there are at most

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

such matrices.