Mathematics 262
Homework 8 solutions

Assignment:

- Section 8.1: 1, 3
- Section 8.2: 2, 3, 6, 9, 13, 17

§8.1, #1: (a) We have

$$(\vec{0}|\beta) = (\vec{0} - \vec{0}|\beta) = (\vec{0}|\beta) - (\vec{0}|\beta) = 0$$

(b) If $(\alpha|\beta) = 0$ for all $\beta \in V$, then in particular, $(\alpha|\alpha) = 0$. Therefore α must be $\vec{0}$.

§8.1, #3: I claim that any inner product on **R** (or on **C**) is determined by the value of $(1|1)$. After all, the inner product of two arbitrary numbers x and y must be $(x|y) = x\overline{y}(1|1)$ by the linearity properties of the inner product. Because $(x|x)$ must be positive whenever x is nonzero, then $(1|1)$ must be a positive real number. Hence every inner product on **R** (or on **C**) is of the form

$$(x|y) = x\overline{y}c$$

for some positive real number c. Conversely, every positive c determines an inner product via this formula.

§8.2, #2: Gram-Schmidt gives $\{(1,0,1),(1,0,-1),(0,3,0)\}$, so normalizing (dividing each vector by its norm) gives the following orthonormal basis: $\{\frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{2}}(1,0,-1),(0,1,0)\}$.

§8.2, #3: Gram-Schmidt gives $\{(1,0,i), \frac{1}{2}(1+i,2,1-i)\}$, so normalizing gives the following orthonormal basis: $\{\frac{1}{\sqrt{2}}(1,0,i), \frac{1}{2\sqrt{2}}(1+i,2,1-i)\}$.

§8.2, #6: First, I'll find an orthonormal basis for W. Since W is one-dimensional, this just corresponds to normalizing the single basis vector: $\alpha = \frac{1}{5}(3,4)$.

(a) The formula for orthogonal projection onto W is

$$\begin{aligned} E(x_1,x_2) &= ((x_1,x_2)\cdot\alpha)\,\alpha \\ &= \frac{1}{5}(3x_1+4x_2)\alpha \\ &= \frac{1}{25}(9x_1+12x_2, 12x_1+16x_2). \end{aligned}$$

(b) So the matrix for E is $\frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

(c) $W^\perp$ is one-dimensional, so I just have to find one vector orthogonal to $(3,4)$. This is easy: $(4,-3)$ is an example, so $W^\perp$ is the subspace spanned by $(4,-3)$.

(d) Let $\beta = \frac{1}{5}(4,-3)$. Then with respect to the orthonormal basis (α,β), the matrix for E is as desired.

§8.2, #9: (a) Suppose that $g(t) = a_0 + a_1t + a_2t^2 + a_3t^3$ is in the orthogonal complement of the subspace of scalar polynomials. Then

$$\int_0^1 cg(t)dt = c\int_0^1 g(t)dt = 0$$

for every scalar c. In particular, $\int_0^1 g(t)dt = 0$. We can evaluate this integral: it is $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}$. So the orthogonal complement is the subspace of all polynomials $g(t) = a_0 + a_1t + a_2t^2 + a_3t^3$ satisfying

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0.$$

(b) Start with $\alpha_0 = 1$. This has norm 1. Then let

$$\begin{aligned}\alpha_1 &= x - (x|\alpha_0)\alpha_0 \\ &= x - (\int_0^1 t dt)1 \\ &= x - \frac{1}{2}.\end{aligned}$$

This has norm $(\int_0^1 (t - \frac{1}{2})^2 dt)^{\frac{1}{2}} = \frac{1}{\sqrt{12}}$. Then let

$$\begin{aligned}\alpha_2 &= x^2 - \frac{(x^2|\alpha_1)}{||\alpha_1||^2}\alpha_1 - (x^2|\alpha_0)\alpha_0 \\ &= x^2 - x + \frac{1}{6}.\end{aligned}$$

This has norm $\frac{1}{\sqrt{180}}$. Finally,

$$\alpha_3 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$$

§8.2, #13: It suffices to show that each element α in S is in $(S^\perp)^\perp$. By definition of $S^\perp$, α is orthogonal to every element in $S^\perp$, and therefore α is in $(S^\perp)^\perp$.

Suppose that V is finite-dimensional, and let W be the subspace spanned by S. Then by one of our theorems, $V = W \oplus W^\perp$, and $W^\perp$ is equal to $S^\perp$, so $V = W \oplus S^\perp$. If we apply the theorem to $S^\perp$, then we find that $V = S^\perp \oplus (S^\perp)^\perp$. Counting dimensions, we find that $\dim W = \dim(S^\perp)^\perp$; since $W \subseteq (S^\perp)^\perp$, then these two subspaces must be equal.

§8.2, #17: I would guess that the even functions (functions $g(t)$ satisfying $g(t) = g(-t)$ for all t) would be the orthogonal complement. Let's see what happens.

If $f(t)$ is odd and $g(t)$ is even, then $f(t)g(t)$ is odd, so

$$(f|g) = \int_{-1}^1 f(t)g(t)dt = 0.$$

Therefore the subspace of even functions is contained in $W^\perp$.

Let $h(t)$ be any function. Then the formula

$$h(t) = \frac{1}{2}(h(t) + h(-t)) + \frac{1}{2}(h(t) - h(-t))$$

displays $h(t)$ as being a sum of an even function and an odd function. For shorthand, write $h(t) = g(t) + f(t)$, where $g(t)$ is even and $f(t)$ is odd. If $h(t) \in W^\perp$, then $(h|f) = 0$. But $(h|f) = (g|f) + (f|f)$. $(g|f) = 0$ because g is even and f is odd. So if $h \in W^\perp$, with $h = g + f$, then $(f|f) = 0$, hence $f(t) = 0$, and hence $h = g$. In other words, if $h(t)$ is in $W^\perp$, then $h(t)$ is even. Therefore, $W^\perp$ is contained in the subspace of even functions.

Therefore $W^\perp$ is equal to the subspace of even functions.

(Alternatively, the formula $h = g + f$ shows that V is the sum of the subspaces of even and odd functions. The only function that is both even and odd is the zero function, and therefore this is actually a direct sum: $V = W \oplus W'$, where W' is the subspace of even functions. We showed that $W^\perp \supseteq W'$; this direct sum decomposition shows that $W^\perp \subseteq W'$. Therefore $W^\perp = W'$.)