Assignment:

- Section 8.1: 1, 3
- Section 8.2: 2, 3, 6, 9, 13, 17

§ 8.1, #1: (a) We have

\[(0|\beta) = (0 - 0|\beta) = (0|\beta) - (0|\beta) = 0\]

(b) If \((\alpha|\beta) = 0\) for all \(\beta \in V\), then in particular, \((\alpha|\alpha) = 0\). Therefore \(\alpha\) must be \(\vec{0}\).

§ 8.1, #3: I claim that any inner product on \(\mathbb{R}\) (or on \(\mathbb{C}\)) is determined by the value of \((1|1)\). After all, the inner product of two arbitrary numbers \(x\) and \(y\) must be \((x|y) = xy(1|1)\) by the linearity properties of the inner product. Because \((x|x)\) must be positive whenever \(x\) is nonzero, then \((1|1)\) must be a positive real number. Hence every inner product on \(\mathbb{R}\) (or on \(\mathbb{C}\)) is of the form \((x|y) = xy\) for some positive real number \(c\). Conversely, every positive \(c\) determines an inner product via this formula.

§ 8.2, #2: Gram-Schmidt gives \(\{(1,0,1), (1,0,-1), (0,3,0)\}\), so normalizing (dividing each vector by its norm) gives the following orthonormal basis:

\(\{(1,0,1), 1/\sqrt{2}(1,0,-1), (0,1,0)\}\).

§ 8.2, #3: Gram-Schmidt gives \(\{(1,0,i), 1/2(1+i,2,1-i)\}\), so normalizing gives the following orthonormal basis:

\(\{1/\sqrt{2}(1,0,i), 1/2\sqrt{2}(1+i,2,1-i)\}\).

§ 8.2, #6: First, I’ll find an orthonormal basis for \(W\). Since \(W\) is one-dimensional, this just corresponds to normalizing the single basis vector: \(\alpha = 1/5(3,4)\).

(a) The formula for orthogonal projection onto \(W\) is

\[E(x_1, x_2) = ((x_1, x_2) \cdot \alpha) \alpha = \frac{1}{5}(3x_1 + 4x_2)\alpha = \frac{1}{25}(9x_1 + 12x_2, 12x_1 + 16x_2).\]

(b) So the matrix for \(E\) is \(\frac{1}{25}\left[\begin{array}{cc} 9 & 12 \\ 12 & 16 \end{array}\right]\).

(c) \(W^\perp\) is one-dimensional, so I just have to find one vector orthogonal to \((3,4)\). This is easy: \((4,-3)\) is an example, so \(W^\perp\) is the subspace spanned by \((4,-3)\).

(d) Let \(\beta = 1/5(4,-3)\). Then with respect to the orthonormal basis \((\alpha, \beta)\), the matrix for \(E\) is as desired.

§ 8.2, #9: (a) Suppose that \(g(t) = a_0 + a_1t + a_2t^2 + a_3t^3\) is in the orthogonal complement of the subspace of scalar polynomials. Then

\[\int_0^1 cg(t)dt = c \int_0^1 g(t)dt = 0\]

for every scalar \(c\). In particular, \(\int_0^1 g(t)dt = 0\). We can evaluate this integral: it is \(a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}\). So the orthogonal complement is the subspace of all polynomials \(g(t) = a_0 + a_1t + a_2t^2 + a_3t^3\) satisfying

\[a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0.\]
(b) Start with $\alpha_0 = 1$. This has norm 1. Then let

$$\alpha_1 = x - (x|\alpha_0)\alpha_0$$
$$= x - (\int_0^1 tdt)1$$
$$= x - \frac{1}{2}.$$

This has norm $(\int_0^1 (t - \frac{1}{2})^2dt)\frac{1}{2} = \frac{1}{\sqrt{12}}$. Then let

$$\alpha_2 = x^2 - \frac{(x^2|\alpha_1)}{||\alpha_1||^2} \alpha_1 - (x^2|\alpha_0)\alpha_0$$
$$= x^2 - x + \frac{1}{6}.$$

This has norm $\frac{1}{\sqrt{150}}$. Finally,

$$\alpha_3 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$$

§8.2, #13: It suffices to show that each element $\alpha$ in $S$ is in $(S^\perp)^\perp$. By definition of $S^\perp$, $\alpha$ is orthogonal to every element in $S^\perp$, and therefore $\alpha$ is in $(S^\perp)^\perp$. Suppose that $V$ is finite-dimensional, and let $W$ be the subspace spanned by $S$. Then by one of our theorems, $V = W \oplus W^\perp$, and $W^\perp$ is equal to $S^\perp$, so $V = W \oplus S^\perp$. If we apply the theorem to $S^\perp$, then we find that $V = S^\perp \oplus (S^\perp)^\perp$. Counting dimensions, we find that $\dim W = \dim(S^\perp)^\perp$; since $W \subseteq (S^\perp)^\perp$, then these two subspaces must be equal.

§8.2, #17: I would guess that the even functions (functions $g(t)$ satisfying $g(t) = g(-t)$ for all $t$) would be the orthogonal complement. Let’s see what happens.

If $f(t)$ is odd and $g(t)$ is even, then $f(t)g(t)$ is odd, so

$$(f|g) = \int_{-1}^1 f(t)g(t)dt = 0.$$

Therefore the subspace of even functions is contained in $W^\perp$.

Let $h(t)$ be any function. Then the formula

$$h(t) = \frac{1}{2}(h(t) + h(-t)) + \frac{1}{2}(h(t) - h(-t))$$

displays $h(t)$ as being a sum of an even function and an odd function. For shorthand, write $h(t) = g(t) + f(t)$, where $g(t)$ is even and $f(t)$ is odd. If $h(t) \in W^\perp$, then $(h|f) = 0$. But $(h|f) = (g|f) + (f|f)$. $(g|f) = 0$ because $g$ is even and $f$ is odd. So if $h \in W^\perp$, with $h = g + f$, then $(f|f) = 0$, hence $f(t) = 0$, and hence $h = g$. In other words, if $h(t)$ is in $W^\perp$, then $h(t)$ is even. Therefore, $W^\perp$ is contained in the subspace of even functions.

Therefore $W^\perp$ is equal to the subspace of even functions.

(Alternatively, the formula $h = g + f$ shows that $V$ is the sum of the subspaces of even and odd functions. The only function that is both even and odd is the zero function, and therefore this is actually a direct sum: $V = W \oplus W'$, where $W'$ is the subspace of even functions. We showed that $W^\perp \supseteq W'$; this direct sum decomposition shows that $W^\perp \subseteq W'$. Therefore $W^\perp = W'$.)