Mathematics 262
Homework 9 solutions

Assignment:

- Section 8.3: 2, 5, 9, 12

§8.3, #2: Since I know what T does to the standard basis vectors, I can write down its matrix: $A = \begin{bmatrix} 1+i & i \\ 2 & i \end{bmatrix}$. Since the standard basis is orthonormal, then the matrix for T^* is the conjugate transpose of this: $A^* = \begin{bmatrix} 1-i & 2 \\ -i & -i \end{bmatrix}$. To check whether T commutes with T^*, just compute AA^* and A^*A, and see if they're the same. I think the results are:

$$AA^* = \begin{bmatrix} 3 & 3+2i \\ 3-2i & 5 \end{bmatrix}, \qquad A^*A = \begin{bmatrix} 6 & 1+3i \\ 1-3i & 2 \end{bmatrix}.$$

So they don't commute. (If you'd rather work with linear operators, you can evaluate $TT^*(\varepsilon_j)$ and $T^*T(\varepsilon_j)$ for $j = 1, 2$, and see if you get the same thing.)

§8.3, #5: The easiest thing to do is to use the properties of the adjoint: if T is invertible, then $I = TT^{-1}$. Take the adjoint of both sides, and use the fact that $I^* = I$:

$$I = I^* = (TT^{-1})^* = (T^{-1})^*T^*.$$

Similarly, the equation $I = T^{-1}T$ yields $I = T^*(T^{-1})^*$. Therefore the operator T^* is invertible, because I have found its inverse: $(T^{-1})^*$. (A linear transformation U is invertible if and only if there is another linear transformation S so that $SU = I$ and $US = I$.)

§8.3, #9: I know that if I can find the matrix for D with respect to an *orthonormal* basis, then it's easy to get the matrix for D^*. Fortunately, as part of a previous assignment, I know such a basis. Actually, the previous assignment gives an orthogonal basis, so I find the norm of each basis vector and divide by it to get the following orthonormal basis:

$$\begin{aligned}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \left(1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6}), 20\sqrt{7}(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20})\right) \\ &= \left(1, \sqrt{3}(2x-1), \sqrt{5}(6x^2 - 6x + 1), \sqrt{7}(20x^3 - 30x^2 + 12x - 1)\right).\end{aligned}$$

To compute the matrix for D, I apply D to each basis element and expand the result in terms of this basis; in other words, as we saw in class, the (i, j)-entry of the matrix for D is $(D\alpha_j|\alpha_i) = \int_0^1 \frac{d\alpha_j}{dt}\alpha_i dt$.

$$\begin{aligned}
D(\alpha_1) &= D(1) = 0. \\
D(\alpha_2) &= 2\sqrt{3}D(x - \frac{1}{2}) = 2\sqrt{3} = 2\sqrt{3}\alpha_1. \\
D(\alpha_3) &= 6\sqrt{5}D(x^2 - x + \frac{1}{6}) = 6\sqrt{5}(2x-1) = 2\sqrt{15}(\alpha_2). \\
D(\alpha_4) &= 20\sqrt{7}D(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}) \\
&= 20\sqrt{7}(3x^2 - 3x + \frac{3}{5}) \\
&= 20\sqrt{7}\left(3(x^2 - x + \frac{1}{6}) - \frac{1}{2} + \frac{3}{5}\right) \\
&= 20\sqrt{7}(3\alpha_3 + \frac{1}{10}\alpha_1) \\
&= 2\sqrt{35}\alpha_3 + 2\sqrt{7}\alpha_1.
\end{aligned}$$

So the matrix for D is

$$\begin{bmatrix} 0 & 2\sqrt{3} & 0 & 2\sqrt{7} \\ 0 & 0 & 2\sqrt{15} & 0 \\ 0 & 0 & 0 & 2\sqrt{35} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the matrix for D^* is the conjugate transpose of this, or actually just the transpose, since we are working with real numbers:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 \\ 2\sqrt{7} & 0 & 2\sqrt{35} & 0 \end{bmatrix}.$$

For example,

$$D^*(1) = D^*(\alpha_1) = 2\sqrt{3}\alpha_2 + 2\sqrt{7}\alpha_4 = 280x^3 - 420x^2 + 180x - 20.$$

§8.3, #12: If T is self-adjoint, then $(T\alpha|\beta) = (\alpha|T\beta)$ for all α, β. In particular, $(T\alpha|\alpha) = (\alpha|T\alpha)$. On the other hand, by switching factors, $(\alpha|T\alpha) = \overline{(T\alpha|\alpha)}$. We conclude that $(T\alpha|\alpha) = \overline{(T\alpha|\alpha)}$ for all α, and therefore $(T\alpha|\alpha)$ is real for all α.

Suppose, conversely, that $(T\alpha|\alpha)$ is real for all α. I want to show that for all α and β,

$$(T\alpha|\beta) = (\alpha|T\beta).$$

I'll do this in sort of a roundabout way. Let z be a complex number, and consider

$$(T(\alpha + z\beta)|\alpha + z\beta) = (T\alpha|\alpha) + \overline{z}(T\alpha|\beta) + z(T\beta|\alpha) + |z|^2(T\beta|\beta).$$

This is a real number (by assumption, using $\alpha + z\beta$ instead of α), as are $(T\alpha|\alpha)$ and $|z|^2(T\beta|\beta)$. (Recall that $|z|^2 = z\overline{z}$ is always a real number.) Therefore $\overline{z}(T\alpha|\beta) + z(T\beta|\alpha)$ is real for every complex number z. I claim that this implies that $(T\alpha|\beta) = \overline{(T\beta|\alpha)}$. If I can verify the claim, then

$$(T\alpha|\beta) = \overline{(T\beta|\alpha)} = (\alpha|T\beta),$$

as desired.

I have two complex numbers, $r = (T\alpha|\beta)$ and $s = (T\beta|\alpha)$, and I know that for every complex number z, $\overline{z}r + zs$ is a real number. I want to show that $r = \overline{s}$. (The idea is that $\overline{zs} + zs$ is real, and $r = \overline{s}$ should be the only solution that works for every z.) This is pretty easy: let $r = a + bi$, and let $s = c + di$. I want to show that $a = c$ and $b = -d$. When $z = 1$, I have

$$\overline{z}r + zs = r + s = (a + c) + i(b + d).$$

Since this is a real number, then $b = -d$. When $z = i$,

$$\overline{z}r + zs = -ir + is = -ia + b + ic - d = (b - d) + i(c - a).$$

Since this is a real number, then $a = c$. This finishes the claim, and hence the problem.

(Alternatively, if I know that $\overline{z}r + zs$ is real for every complex number z, then it is equal to its complex conjugate. For $z = 1$ and $z = i$, this gives:

$$r + s = \overline{r} + \overline{s},$$
$$-ir + is = i\overline{r} - i\overline{s}.$$

Multiply the second equation by i:

$$r - s = -\overline{r} + \overline{s}.$$

Now add it to the first equation:

$$2r = 2\overline{s}.$$

Dividing by 2 gives the desired result.)