Exam 1

Friday the 13th (of February)

General instructions: You may not use calculators. Unless otherwise specified, treat all matrices (etc.) as being defined over the real numbers $\mathbf{R}$. You may use any result from class or from the homework, if you want to. If I ask a question about $n \times n$ matrices and you're not sure how to do it, you will get partial credit if you just work on the 2×2 case or the 3×3 case. (You might even get a tiny bit of credit for the 1×1 case.)

Short answer

(5) 1. Give a formula for the *determinant* of an $n \times n$ matrix.

(5) 2. Suppose that T is a linear operator on an n-dimensional vector space. Give the definition of an *eigenvalue* of T.

(5) 3. Describe *Cramer's rule.*

True or false: For the next two problems, tell me whether each statement is true or false. If it's true, give a *brief* reason why. If it's false, give a counterexample.

(10) 4. If A is an $n \times n$ matrix with rank less than n, then 0 is an eigenvalue of A.

(10) 5. For $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$.

Computations

(10) 6. (a) Compute the determinant of $\begin{bmatrix} 1 & -2 & 3 & -4 \\ 5 & 6 & -7 & -8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix}$.

(10) (b) Compute the inverse of $\begin{bmatrix} 3 & 0 & 0 \\ -5 & -2 & 0 \\ 4 & 0 & 1 \end{bmatrix}$.

7. Find the eigenvalues of the following matrices, and for each eigenvalue, find a basis for the corresponding eigenspace. Is each matrix diagonalizable? (Give a brief reason.)

(10) (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(10) (b) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (Your answer may depend on θ.)

Theory

(15) 8. (a) Suppose that A is a 2×2 matrix of real numbers which is symmetric (i.e., $A = A^t$). Prove that A is similar to a diagonal matrix.

(b) **(extra credit)** Can you extend this result to 3×3 symmetric matrices, or in general to $n \times n$ symmetric matrices?

(10) 9. (a) Let A be an $n \times n$ matrix with characteristic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

What is the constant term a_0, in terms of A? (Hint: plug $x = 0$ into the characteristic polynomial.)

(b) **(extra credit)** Can you express the coefficient a_{n-1} of x^{n-1} in a similar way? What about the other coefficients?