Exam 2

Due by beginning of class, Wednesday, April 22

General instructions: This is an *open book and open notes* exam. Of course, you can't discuss the exam with anyone except me. You may not use calculators or computers. (If you can figure out a helpful way to use a slide rule for any of these problems, you are welcome to do so.)

Part 1. You must do this problem.

1. Let $O(n)$ denote the set of $n \times n$ orthogonal matrices (with real entries).
   (a) Show that every matrix in $O(n)$ has determinant equal to 1 or -1.

   $SO(n)$ is the subset of $O(n)$ of all orthogonal matrices with determinant one. Matrices in $SO(n)$ are called "special orthogonal" matrices.

   (b) Describe the sets $SO(1)$ and $SO(2)$. (In other words, tell me all of the matrices contained in each of these sets.)

   (c) Suppose that $A \in SO(3)$. Assume that $a_{13} \neq 0$. Show that the values of a_{11}, a_{12}, a_{21}, and a_{22}, together with the sign of a_{13}, determine the rest of the entries of A. (If you have this much information about a special orthogonal matrix, explain how to find all of the entries.)

   (d) **(extra credit)** Give examples to show that you need at least this much information (if you knew less, show that you can't uniquely determine all of the entries of the matrix).

   (e) **(extra credit)** The set of $n \times n$ "special unitary" matrices, $SU(n)$, is the set of all unitary matrices (complex entries) with determinant one. Figure out as much as possible about $SU(2)$.

Part 2—material from Chapter 6. Do two of the following five problems.

2. Consider the matrix $A = \begin{bmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{bmatrix}$. Show that the minimal polynomial of A is $x^3 + ax^2 + bx + c$.

3. Do problem 13 in Section 6.4 (p. 206).

4. Do problem 3 in Section 6.5 (p. 208).

5. Let $A = \begin{bmatrix} 1 & \frac{1}{2} \\ y & 3 \end{bmatrix}$.
   (a) For which complex numbers y is A similar (over $\mathbf{C}$) to an upper triangular matrix?
   (b) For which complex numbers y is A similar (over $\mathbf{C}$) to a diagonal matrix?
   (c) For which real numbers y is A similar (over $\mathbf{R}$) to a diagonal matrix?

6. (a) Find a 3×3 matrix A so that
   - 1 is an eigenvalue of A, with eigenvector $(0, 0, 1)$,
   - 5 is an eigenvalue of A, with eigenvector $(\frac{3}{5}, \frac{4}{5}, 0)$, and
   - -10 is an eigenvalue of A, with eigenvector $(\frac{4}{5}, -\frac{3}{5}, 0)$.

   (b) Indeed, if A is $n \times n$ and satisfies the equations $A\alpha_i = c_i\alpha_i$ for $i = 1, \dots, n$, tell me how to find A.

Part 3—material from Chapter 8. Do two of the following three problems.

7. Let V denote the vector space of all polynomials with real coefficients, of degree at most *two*; put the usual inner product on it:
$$(f|g) = \int_0^1 f(x)g(x)dx.$$
Based on what we did on a homework problem, we know that if we apply the Gram-Schmidt procedure to the basis $\{1, x, x^2\}$, we get this orthonormal basis:
$$\{1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\}.$$
(In the homework problem, we were working with polynomials of degree at most three, so we had a fourth polynomial, in addition.)

   (a) Apply Gram-Schmidt to the basis $\{x^2, x, 1\}$ to get another orthonormal basis for V (note: not just orthogonal, but orthonormal).

   (b) Use either of these orthonormal bases to find a polynomial $g(x)$ so that
$$\int_0^1 f(x)g(x)dx = f'(0)$$
for every polynomial $f \in V$. (Hint: the function that sends a polynomial $f(x)$ to the number $f'(0)$ is a linear functional on V.)

8. Let W_1 and W_2 be the following subspaces of $\mathbf{R}^3$:
$$W_1 = \text{Span}((1, 1, 1), (1, 0, 1)),$$
$$W_2 = \text{Span}(0, 0, 1).$$

   (a) Show that $\mathbf{R}^3 = W_1 \oplus W_2$.

   (b) Find an inner product on $\mathbf{R}^3$ such that $W_1 = W_2^{\perp}$. If you can't work out the details for some reason, explain the procedure for solving the problem.

9. Let $A = \begin{bmatrix} a & \frac{1}{2} \\ b & -\frac{\sqrt{3}}{2} \end{bmatrix}$.

   (a) For which complex numbers a and b is A unitary?

   (b) For which complex numbers a and b is A self-adjoint?

   (c) (**extra credit**) For which complex numbers a and b is A normal?