Math 262 – Linear Algebra
Spring 2001 (Exam 1)

1. *(25 points)* Find a 3×3 matrix A over the reals so that (i)–(iii) below all hold:
(i) 1 is an eigenvalue of A, with eigenvector $(0, 0, 1)$,
(ii) 5 is an eigenvalue of A, with eigenvector $(\frac{3}{5}, \frac{4}{5}, 0)$, and
(iii) -10 is an eigenvalue of A, with eigenvector $(\frac{4}{5}, -\frac{3}{5}, 0)$.

2. *(25 points)* Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (x, x + 2y + 3z, x + 2z).$$

a) Write down the matrix (with respect to the standard ordered basis) of T and determine the characteristic polynomial of T.

b) Find the eigenvalues of T.

c) Find a basis for the eigenspace of T corresponding to the *double* eigenvalue.

d) Is T diagonalizable? Explain.

3. *(25 points)* Prove the *parallelogram law* on an inner product space V; that is, show that for all vectors x and y in V

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$

What does this equality say about parallelograms in $\mathbb{R}^2$?

4. *(25 points)* Consider the real inner product space V consisting of polynomials of degree at most two over the real numbers with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$.

a) Apply the Gram-Schmidt process to the ordered set of vectors $X = \{x, x^2\}$ in V. Write down an orthonormal basis for the span W of X.

b) Determine the orthogonal projection of the constant function 1 on W.

c) Find a basis for the orthogonal complement $W^{\perp}$ of W.