Math 262, Exam 2

Closed book, take home exam. Calculators may be used.

1 *(25 points)* Consider the following 2×2 matrix A over the complex numbers

$$A = \begin{pmatrix} a & \frac{1}{2} \\ b & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

(a) Write down the adjoint A^* of A.

(b) For which complex numbers a and b is A unitary?

(c) For which complex numbers a and b is A self-adjoint?

(d) For which complex numbers a and b is A normal?

2. *(25 points)*

(a) State the spectral theorem for normal matrices over $\mathbf{C}$.

(b) Determine the complex eigenvalues and corresponding eigenspaces in $\mathbf{C}^2$ of

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbf{C})$$

where $\theta \in \mathbf{R}$ is not a multiple of π.

(c) Compute the matrices (with respect to the standard basis of $\mathbf{C}^2$) of the orthogonal projections on the eigenspaces of A (with respect to the standard dot product on $\mathbf{C}^2$). Verify the spectral theorem for A.

(d) Is there a real orthogonal matrix Q such that Q^*AQ is diagonal?

3. *(25 points)* Let $T\colon V \to V$ be a linear operator on a finite-dimensional complex inner product space V.

(a) Show that if T is self-adjoint, then $\langle Tx, x\rangle$ is a real number for all $x \in V$.

(b) Show that if $\langle Tx, x\rangle = 0$ for all $x \in V$, then $T(y) = 0$ for all $y \in V$ (Hint: use that $\langle T(y+z), (y+z)\rangle = 0$ and $\langle T(y+iz), (y+iz)\rangle = 0$ for any $y, z \in V$ where $i = \sqrt{-1}$)

(c) Show that if $\langle Tx, x\rangle$ is real for all $x \in V$, then T is self-adjoint.

4. *(25 points)* Let $O(n)$ denote the set of $n \times n$ orthogonal matrices (with real entries).

(a) Show that every matrix in $O(n)$ has determinant equal to 1 or -1.

(b) $SO(n)$ is defined to be the subset of $O(n)$ of all orthogonal matrices with determinant one. Matrices in $SO(n)$ are called "special orthogonal" matrices. Describe the sets $SO(1)$ and $SO(2)$. (In other words, tell me all of the matrices contained in each of these sets.)

(c) Suppose that $A \in SO(3)$. Assume that $a_{13} \neq 0$. Show that the values of a_{11}, a_{12}, a_{21}, and a_{22}, together with the sign of a_{13}, determine the rest of the entries of A. (If you have this much information about a special orthogonal matrix, explain how to find all of the entries.)

Bonus Question The set of $n \times n$ "special unitary" matrices, $SU(n)$, is the set of all unitary matrices (complex entries) with determinant one. Find the simplest description you can of the elements of $SU(2)$.