Math 262, Spring 2001; Final Exam

This is a closed book exam. Calculators may be used. All working must be shown to receive full points.

1. *(25 points)* Consider the following 2×2 matrices A, B, C, D over the real numbers:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

a) Which of the matrices A–D are orthogonally similar over the real numbers to a diagonal matrix?

b) Which of the matrices A–D are unitarily similar over the complex numbers to a diagonal matrix?

c) Which of the matrices A–D are similar over the complex numbers to a diagonal matrix?

2. *(25 points)* Prove that for any vectors x, y in a real inner product space V,

$$\langle x, y \rangle = \frac{1}{4}(\, ||x + y\,||^2 - ||x - y\,||^2\,)$$

What does this tell you about a parallelogram in $\mathbf{R^2}$ for which the two diagonals are of equal length?

3. *(25 points)* Consider the space $M_{n\times n}(\mathbf{R})$ of all $n \times n$ real matrices endowed with the inner product $\langle A, B\rangle$ = trace of B^*A (you may assume that this is an inner product). Consider the subset W of $M_{n\times n}(\mathbf{R})$ consisting of those matrices of zero trace. Show that W is a subspace of dimension $n^2 - 1$ and find its orthogonal complement.

4. *(25 points)* a) A linear operator T on a 7 dimensional complex vector space V satisfies the following conditions: $\text{rank}(T + 2) = 5$, $\text{rank}((T + 2)^2) = 3$, $\text{rank}((T + 2)^3) = 2$, $\text{rank}(T - 3) = 6$, and $\text{rank}((T - 3)^2) = 5$. Write down the Jordan canonical form of T.

b) Write down the possible Jordan canonical forms of a linear operator S on V with characteristic polynomial $-(t - 2)^4t^3$ and minimal polynomial $(t - 2)^2t^2$.

5. *(25 points)* A linear operator N is called nilpotent if $N^m = 0$ for some $m > 0$.

a) Show that a nilpotent operator N on a finite-dimensional space V has a Jordan canonical form.

b) Show that $\text{Id}_V - N$ is invertible if N is nilpotent.

6. *(25 points)* a) Let $p(t)$ denote the minimal polynomial of a linear operator T on a finite-dimensional vector space over a field F. If $q(t) \in F[t]$ is a polynomial with $q(T) = T_0$ (the zero operator), what is the relation between $p(t)$ and $q(t)$?

b) State necessary and sufficient conditions in terms of $p(t)$ for diagonalizability of T.

c) Suppose that $A \in \text{Mat}_{n\times n}(\mathbf{C})$ satisfies $A^m = \text{Id}_n$ for some $m \geq 1$. Use (a) and (b) to show that A is similar to a diagonal matrix.

d) Give an example of a matrix $B \in \text{Mat}_{2\times 2}(\mathbf{Z}_2)$ satisfying $B^2 = \text{Id}_2$ but such that B is not similar to a diagonal matrix.