UNIVERSITY OF NOTRE DAME
DEPARTMENT OF MATHEMATICS

MATH 318 Final Examination December 19, 2002

Name: ______________________________

The Honor Code is in effect
The format is "open−book" and "open−notes"
The time limit is 2 hours

Prob. 1	/ 12
Prob. 2	/ 08
Prob. 3	/ 10
Prob. 4	/ 10
Prob. 5	/ 10
Total	/ 50

1. In a performance test, a car starting from rest travels 1320 feet in 16.0 seconds and attains a velocity of 126.0 feet / second in doing so. The objective of this problem is to estimate by the technique described below the time τ it takes for the car to reach a velocity of 88 feet / second.

(a) [6 points] Let x be the distance the car travels in time t. Also let x' (velocity), x'' (acceleration), and x''' (jerk) be respectively the 1st, 2nd, and 3rd derivatives of x with respect to time. Our given problem data are:

$$x = x' = 0 \text{ when } t \text{ is } 0; \quad x = 1320 \text{ and } x' = 126.0 \text{ when } t \text{ is } 16.0.$$

Write Taylor expansions (up to and including the third−derivative term) for x and x' at (t = 16.0) in terms of x and its derivatives at (t = 0) as needed. Solve the resulting equations for x'' and x''' at (t = 0) and **report your results to at least six significant digits**.

(b) [6 points] Use the results of part (a) to express the target velocity of 88 ft / s at time τ as a Taylor expansion (up to and including the third−derivative term) in terms of x derivatives at (t = 0). Solve the resulting quadratic for τ. **Specify clearly which root is τ and give your answer to at least two decimal places**.

2. In a performance test, a car starting from rest travels 1320 feet in 16.0 seconds and attains a velocity of 126.0 feet / second in doing so. The objective of this problem is to estimate by the technique described below the time τ it takes for the car to reach a velocity of 88 feet / second.

(a) [4 points] Let t be the time it takes for the car to reach a velocity v. A model for the distance the car travels may be expressed in the form

$$1320 \text{ ft} = (16.0 \text{ s})(126.0 \text{ ft/s}) - \int_0^{126} t \, dv$$

Apply Simpson's rule with two intervals to the model above, and thus estimate the time t for the car to reach a velocity of 63.0 ft / s. **Please give your answer to at least six significant digits**.

(b) [4 points] Use t values at v equal to 0, 63 (from part (a)), and 126 ft / s to obtain by polynomial interpolation the time τ at which the car's velocity is 88 ft / s. In doing so, t is to be treated as a second−degree polynomial in v − that is, v is the independent variable. **Please give your answer to at least two decimal places**.

3. A function $y(x)$ has a first derivative denoted by y', and it satisfies the ordinary differential equation and initial condition

$$y' = f(x, y) = y/(2x + y^2); \qquad y(0) = 1$$

(a) [4 points] Start from the initial condition and use one step of size ($h = 0.1$) to estimate $y(0.1)$ by the two−stage, second−order Runge−Kutta method

$$y(0.1) = y(0) + h(k_1/3 + 2k_2/3)$$

in which k_1 and k_2 are the stages, and k_1 is $f(0, 1)$. **Please show your work, and report your result to 5 significant digits**.

(b) [6 points] Use one step of size ($h = 0.1$) to estimate $\zeta = y(0.1)$ by the second−order implicit method

$$\zeta = 1 + [h/2][f(0, 1) + f(h, \zeta)]$$

Your result for ζ must be sufficiently accurate so that the left and right sides of the equation above differ by no more than 0.00001 in magnitude. **Please show basic steps of your work, and report your result to 5 significant digits**.

4. [10 points] A function $y(x)$ satisfies the ordinary differential equation and initial conditions

$$(x^2 y) y'' + 2y' = 0; \qquad y(2) = 0.5, \quad y'(2) = -0.25$$

in which y' and y'' are the first and second derivatives, respectively, of y with re−spect to x.

Denote y' by z, and recast the differential equation as a system of two first−order ordinary differential equations in y and z. Then use one step of the modified Euler method to estimate y and z when x is equal to 2.2. **Please show your work, and report both results to 5 significant digits**.

5. Consider the matrix below in which k is **positive**:

$$\mathbf{A} = \begin{bmatrix} k & -8 \\ -8 & 4k \end{bmatrix}$$

(a) [4 points] Determine k such that one eigenvalue of $\mathbf{A}$ is 5 times the other; that is, if λ is the smaller eigenvalue, the larger one is 5λ. **Please show your work** and give your result for k **to three decimal places**.

(b) [6 points] When k is 10, an eigenvector $\mathbf{q}$ associated with the **smaller** of the two eigenvalues of $\mathbf{A}$ is

$$\mathbf{q} = \begin{bmatrix} a \\ b \end{bmatrix}; \quad a > 0, \;\; a^2 + b^2 = 1$$

Compute the component a **to three decimal places**. **Please show your work**.