If $r = \rho$ is a double eigenvalue of the 2 x 2 matrix A with just one linearly independent eigenvector ξ and η is a vector satisfying $(A - \rho I)\eta = \xi$, then the general solution of $x' = Ax$ is:

$c_1 e^{\rho t}\xi + c_2[te^{\rho t}\xi + e^{\rho t}\eta]$

$c_1 e^{\rho t}\xi + c_2 te^{\rho t}\xi + \eta$ $c_1 e^{\rho t}\xi + c_2 e^{\rho t}\eta$ $c_1 e^{\rho t}\xi + c_2 e^{\rho t}\eta$ $c_1 e^{\rho t}\xi + c_2 e^{\rho t}\eta$

The general solution of $x' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$ is

$c_1 \begin{pmatrix} \cos\ t \\ -\sin\ t \end{pmatrix} + c_2 \begin{pmatrix} \sin\ t \\ \cos\ t \end{pmatrix}$

$c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$c_1 \begin{pmatrix} \cos\ t \\ \cos\ t \end{pmatrix} + c_2 \begin{pmatrix} \sin\ t \\ \sin\ t \end{pmatrix}$

$c_1 \begin{pmatrix} e^t \cos\ t \\ -e^t \sin\ t \end{pmatrix} + c_2 \begin{pmatrix} e^t \cos\ t \\ e^t \sin\ t \end{pmatrix}$

$c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Assume x' = Ax has a fundamental matrix $\Psi(t) = \begin{pmatrix} (1+t) & (2-t) \\ (t-2) & (1+t) \end{pmatrix}$. Note that $\Psi(0) \neq I$. Assume $x(t)$ is the solution of the initial value problem $x' = Ax$ and $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Calculate $x(t)$ at $t = 2$.

$x(2) = \frac{3}{5} \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

$x(2) = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$

$x(2) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$x(2) = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$

$x(2) = \frac{3}{7} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$ has eigenvalues 1 and -3 with corresponding eigenvectors $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Now consider the nonhomogeneous equation $x' = Ax + g(t)$ where A is the above matrix and $g(t) = \begin{pmatrix} 0 \\ -2e^{-2t} \end{pmatrix}$.

Using the above information and the method of variation of parameters with $X_p = \Psi U$, the equations satisfied by the components u_1' and u_2' of U' are

$$\begin{aligned} e^t u_1' + e^{-3t} u_2' &= 0 \\ e^{-3t} u_2' &= e^{-2t} \end{aligned}$$

$$\begin{aligned} e^{-t} u_1' + e^{-3t} u_2' &= 0 \\ e^{-3t} u_2' &= -2e^{-2t} \end{aligned}$$

$$\begin{aligned} e^t u_1' + e^{-3t} u_2' &= 0 \\ 3e^{-3t} u_2' &= 0 \end{aligned}$$

$$\begin{aligned} u_1' + e^{2t} u_2' &= 0 \\ e^{-2t} u_2' &= -2e^{2t'} \end{aligned}$$

$$\begin{aligned} u_1' + u_2' &= e^t \\ e^{-3t} u_2' &= -2e^{-2t} \end{aligned}$$