**IV.To solve the non-homogeneous equation**

$$(1) \quad y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x) = g(x)$$

**We**

**1. find a fundamental system of solutions** $y_1, y_2 \ldots, y_n$ **of the homogeneous equation;**

**2. find a particular solution** $y_p$ **(any solution at all) of the non-homogeneous equation,**

**3. write the general solution as**

$$y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x).$$

**Every solution of (1) is obtained by a unique choice of the numbers** $c_1, c_2, \ldots, c_n$**.**

**Method of Variation of Parameters**

**We describe this technique for the cases** $n = 1, n = 2$

$\mathbf{n = 1}$ $y' + py = g.$

**Let** $y_1$ **satisfy the homogeneous equation** $y_1' + py_1 = 0$

**Set** $\boxed{y_p = u(x)y_1(x)}$**,** $\quad y' = u'y_1 + uy_1'$

**Substitute in the given equation**

$u'y_1 = uy_1' + puy_1 = g$

$u'y_1 + u(y_1' + py_1) = g$

**or** $u'y_1 = g$ **since** $y_1' + py_1 \equiv 0$**.**

$u = \displaystyle\int \frac{g(x)}{y_1(x)}dx$ **so** $y = uy_1$ **gives**

$y_p(x) = y_1(x)\displaystyle\int \frac{g(x)}{y_1(x)}dx.$

**ex.** $y' - 2y = x$**;** $\quad y_1 = e^{2x}$

$y_p = ue^{2x}$**,**

$y_p' = u'e^{2x} + 2ue^{2x}$

**Therefore,** $u'e^{2x} + 2ue^{2x} - 2ue^{2x} = x$ **or** $u'e^{2x} = x$**; and** $u' = xe^{-2x}$

**Thus** $u = \displaystyle\int xe^{-2x}dx$ **so that (since** $y_p = uy_1$**)**

$$y_p = e^{2x} \int xe^{-2x}dx.$$

**The general solution of the given equation is**

$$y(x) = ce^{2x} + e^{2x} \int xe^{-2x}dx.$$

**<u>n = 2</u>**

$$y'' + py' + qy = g(x)$$

**Let** $y_1$ **and** $y_2$ **be a fundamental system of solutions of the homogeneous equation so that the general solution of the homogeneous equation is**

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

.

**Now set** $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ **where** $u_1$ **and** $u_2$ **are unknown functions:**

**(1) Differentiating, we have**

$$y_p' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'.$$

**(2)Now we set** $\boxed{u_1' y_1 + u_2' y_2 = 0}$

**therefore,** $y_p' = u_1 y_1' + u_2 y_2'$.

**Differentiating again, we have**

**(3)** $y_p'' = u_1' y_1'' + u_2 y_2'' u_1 y_1'' + u_2 y_2''$

**(4) Substitute in the original differential equation**

$$u_1'(y_1'') + u_2'(y_2'') + u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) \equiv g$$

**Since** $y_1$ **and** $y_2$ **satisfy the homogeneous equation, we have**

**(5)** $$\boxed{u_1' y_1 + u_2' y_2 = g(x)}$$

**We rewrite the boxed in equations (2) and (5)**

$$(\mathbf{6}) \qquad (*)\begin{cases} y_1, u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = g \end{cases}$$

**Since $y_1$ and $y_2$ (as well as g) are known, we have two equations for $u_1'$ and $u_2'$. These have a unique solution if and only if the determinant of the coefficients is not zero, i.e.,**

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

**But since $y_1$ and $y_2$ form a fundamental system (and p, q, and g are assumed continuous) this is the Wronskian determinant which dose not vanish. Therefore, equation (6) can be solved for $u_1'$ and $u_2'$ which in turn can be integrated and used to yield $y_p = u_1y_1 + u_2y_2$.**

**If we denote the matrix**

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \text{by W}$$

**and** $\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$ **and** $\begin{pmatrix} 0 \\ g \end{pmatrix}$ **by $U'$ and $G$ respectively, we can rewrite (6) as**

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix},$$

**or as** $\boxed{WU' = G}$

<u>**Ex**</u> $y'' - y = x^2 \qquad g(x) = x^2$
**solve $y'' - y = 0$ to obtain $y_1 = e^x, y_2 = e^{-x}$**
**Set $y_p = u_1y_1 + u_2y_2$**
**Then $u_1'$ and $u_2'$ satisfy**

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$$i.e. \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}.$$

**Thus,**
$e^x\ u_1' + e^{-x}m_2' = 0$
$e^x\ u_1' - e^{-x}u_2' = x^2$

**adding, we find** $2e^x u_1' = x^2$ **therefore** $m_1' = \frac{1}{2}e^{-1}x^2$**,** $u_1' = \frac{1}{2}e^{-x}x^2$**, and** $u_1 = \frac{1}{2}\int x^2 e^{-x}dx_1$

**subtracting, we find** $2e^{-x}u_2' = -x^2$ **therefore** $u_2' = -\frac{1}{2}e^x x^2$ **and** $u_2 = -\frac{1}{2}\int e^x x^2 dx$

**Therefore, the general solution is**

$$y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{2}\int x^2 e^{-x}dx - \frac{1e^{-x}}{2}\int x^2 e^x dx.$$

**For** $n \geq 3$ **the same procedure will work.**

1. **get a fundamental system** $y_1, y_2, \ldots y_n$ **for the homogeneous equation.**
2. **Set** $y_p = u_1 y_1 + u_2 y_2 + \ldots u_n g_n$**.**
3. $u_1', u_2' \ldots u_n'$ **satisfy** $WU' = G$

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}$$

**<u>Ex</u>** $D^2(D-1)y = \sqrt{x}$ **Solve** $D^2(D-1)y = 0$ **to get** $y_0 = c_1 + c_2 x + c_3 e^x$
**Set** $y_p = u_1 + u_2 x + u_3 e^x$**.**
**then**

$$W = \begin{pmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{pmatrix}$$

**and** $u_1', u_2', u3'$ **satisfy**

$$\begin{pmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{x} \end{pmatrix},$$

$$u_1' + xu_2' + e^x u_3' = 0 \Rightarrow u_1' = -xu_2' - e^x u_3' = x^{\frac{3}{2}} - \sqrt{x}$$

$$u_2' + e^x u_3' = 0 \Rightarrow u_2' = -e^x u_3' = -\sqrt{x}$$

$$e^x u_3' = \sqrt{x} \Rightarrow u_3' = e^{-x}\sqrt{x}$$

$$\begin{array}{ll||l||l} u_3' = e^{-x}\sqrt{x} & \text{therefore} & u_2' = -\sqrt{x} = -x^{\frac{1}{2}} & u_1' = x^{\frac{3}{2}} - x^{\frac{1}{2}} \\ u_3 = \int \sqrt{x}e^{-x} & & u_2 = \frac{-2}{3}x^{\frac{3}{2}} & u_1 = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} \end{array}$$

**therefore** $y_p = \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + e^x\int \sqrt{x}e^{-x}dx$
$y_p = \frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + e^x\int \sqrt{x}e^{-x}dx$
**The general solution is**

$$y = c_1 + c_2 x + c_3 e^x + \frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + e^x\int \sqrt{x}e^{-x}dx$$