MULTIPLE CHOICE (6 points each)

1. The characteristic equation of a certain fourth order linear homogeneous differential equation is $(r^2 + 4)\,(r + 2)^2 = 0$.

One of the following functions is a solution. Which one?

a. $3e^{-2x} + x\,e^{-2x}$
b. $3e^{2x} + xe^{-2x}$
c. $3e^{-x} + x^2\,e^{-x}$
d. $3xe^{-x} + e^{-2x}$
e. $3e^{2x} + e^{-2x}$

2. Find the Laplace transform of

$$f(t) = \delta(t - \pi) \cos t \,.$$

a. $-\,e^{-\pi s}$
b. $-\,(\cos \pi)\,e^{-\pi s}$
c. $(\cos \pi)\,e^{-s}$
d. e^{-s}
e. $-e^{\pi s}$

3. Find the inverse Laplace transform of $\dfrac{e^{-2s}}{s^2 - 2s + 2}$.

a. $u_2(t)\, e^{(t-2)} \sin(t-2)$

b. $u_{-2}(t)\, e^t \sin t$

c. $u_2(t-2)\, e^t \sin t$

d. $u_2(t)\, e^t \sin(t+2)$

e. $e^{2t} \sin(t-2)$

4. Solve the initial value problem

$$y'' + y = 1 - u_{\frac{\pi}{2}}(t)$$

$$y(0) = 0$$

$$y'(0) = 1$$

a. $1 - \cos t + \sin t - u_{\frac{\pi}{2}}(t)\,(1 - \sin t)$

b. $1 + \cos t - \sin t - u_{\frac{\pi}{2}}(t)\,(1 - \sin t)$

c. $1 - \cos t - \sin t - u_{\frac{\pi}{2}}(t)\,(\frac{\pi}{2} - \sin t)$

d. $1 + \cos t - \sin t - u_{\frac{\pi}{2}}(t)\,(\frac{\pi}{2} - \sin t)$

e. $1 + \cos t - \sin t - u_{\frac{\pi}{2}}(t)$.

5. Consider the problem

$$y' = y + t$$
$$y(0) = 1$$

Use the Euler method with step size $h = 0.1$ to get an approximation y_1 to the solution at $t = 0.1$ and an approximation y_2 to the solution at $t = 0.2$. Then y_1 and y_2 are given by:

a. $y_1 = 1.10$, $y_2 = 1.22$
b. $y_1 = 1.10$, $y_2 = 1.11$
c. $y_1 = 1.010$, $y_2 = 1.012$
d. $y_1 = 1.10$, $y_2 = 1.02$
e. $y_1 = 1.11$, $y_2 = 1.12$

6. $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ has just one eigenvalue 0 with just one independent eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Which of the following is a solution to the system $x' = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} x$?

a. $t \begin{pmatrix} 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

b. $e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c. $2e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

d. $t \begin{pmatrix} 0 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

e. $t \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

7. The system $x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$ has a complex solution $e^{it} \begin{pmatrix} 2 + i \\ 1 \end{pmatrix}$.

Which of the following is a real solution of this system?

a. $\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$ b. $\begin{pmatrix} -\sin t \\ 2\cos t \end{pmatrix}$ c. $\begin{pmatrix} 5\cos t \\ \sin t \end{pmatrix}$

d. $\begin{pmatrix} 5\sin t \\ \cos t - \sin t \end{pmatrix}$ e. $\begin{pmatrix} \sin t \\ \cos t + 2\sin t \end{pmatrix}$

8. $\left\{ \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}, \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} \right\}$ is a fundamental system of solutions for $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$. Let

$x = u_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + u_2 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ be a solution to $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix}$. Which of the following is the system of equations satisfied by u'_1 and u'_2 ?

a. **Error!**) b. **Error!**)

c. **Error!**) d.**Error!**)

e. $\begin{matrix} e^{-t} u_1' + e^{3t} u_2' = t \\ e^{3t} u_1' - e^{-t} u_2' = e^t \end{matrix}$

9. What is the type of the critical point of $x' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} x$?

a. Proper node, unstable b. Spiral point, unstable

c. Improper node, stable

d. Saddle point, unstable

e. Center, stable

10. What is the type of the critical point of $x' = \begin{pmatrix} -2 & 9 \\ 0 & 3 \end{pmatrix} x$?

a. Saddle point, unstable

b. Spiral point, unstable

c. Proper node, unstable

d. Improper node, stable

e. Center, stable

11. The nonlinear system $\begin{cases} x' = y - e^{-x} \\ y' = y - \cos x \end{cases}$ has a critical point at (0,1) . The corresponding linear system is:

a. $\begin{cases} x' = x + y \\ y' = y \end{cases}$ b. $\begin{cases} x' = x \\ y' = x + y \end{cases}$ c. $\begin{cases} x' = (y - 1) - e^{-x} \\ y' = (y - 1) - \cos x \end{cases}$

d. $\begin{cases} x' = 0 \\ y' = 1 \end{cases}$ e. $\begin{cases} x' = x + y \\ y' = y + x \end{cases}$

12. The system $\begin{cases} x' = x + 3\,y + f\,(x,y) \\ y' = 2\,y + g\,(x, y) \end{cases}$ has a critical point at (0, 0) and the corresponding linear system for this critical point is $\begin{cases} x' = x + 3\,y \\ y' = 2\,y \end{cases}$. Only one of the following could then hold. Which one?

a. $f(x, y) = x^3$, $g(x, y) = y^3$

b. $f\,(x, y) = 1$, $g(x, y) = 2$

c. $f(x, y) = x^2 + y^2$, $g(x, y) = -3\,y$

d. $f(x,y) = -\,2 \sin x$, $g(x, y) = 0$

e. $f(x, y) = x\,y$, $g(x, y) = x$

13. The function $f\,(x) = 3 \tan \frac{5\pi\,x}{2}$ is periodic. Its fundamental (smallest) period is

a. $\frac{2}{5}$ b. $\frac{4}{5}$ c. $\frac{5}{2}\,\pi$ d. $\frac{5}{4}\,\pi$ e. $\frac{1}{5}$

14. Consider the following five functions of x:

$f(x) = 1 + |\sin x|$; $g(x) = \sin x \cos x$; $h(x) = 2e^{2x}$; $s(x) = \ln (x^2 + 1)$;

$T(x) = \sin (x + 1)$.

Only one of the following statements concerning the odd or even nature of these functions is correct. Which one?

a. g is odd and s is even

b. h is even and T is odd

c. f is odd and T is neither even nor odd

d. g is odd and T is odd

e. f is even and s is odd

15. Let f be defined on the interval $-1 \le x \le 1$ by

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{for } -1 \le x < 0 \\ 1 - x^2 & \text{for } 0 < x \le 1 \end{cases} \qquad \text{and} \quad f(x + 2) = f(x)$$

To what value does the Fourier series of f(x) converge at x = 0 ?

a. $\frac{3}{4}$ b. $\frac{-1}{2}$ c. 0 d. $\frac{1}{4}$ e. $\frac{2}{3}$

16. Let f(x) be an odd function on the interval $-\pi/2 < x < \pi/2$. If f (x) = cos x when x > 0, find the value of f (x) when x < 0 .

a. $-\cos x$ b. $\cos x$ c. $\cos(-x)$ d. $-\cos(x + \pi)$

e. $\cos(x - \pi/2)$

17. The value of the coefficient b_9 in the sine series $\sum_{1}^{\bullet} b_n \sin nx$ with period 2π for the function $f(x) = 6$ for $0 \le x \le \pi$, is

a. $\frac{8}{3\pi}$ b. 0 c. $\frac{3}{\pi}$ d. $\frac{\pi}{6}$ e. $\frac{6}{\pi}$

18. A metal bar of length 100 cm is heated with one end held at 20º C and the other held at 70º C. After a long time (i,.e. $t \rightarrow \infty$) the temperature distribution in the bar is approximately given by

a. $\frac{x}{2} + 20$

b. 0 everywhere

c. 45 everywhere

d. $4x + 70$

e. $\frac{x}{25} + 20$

19. A metal rod 100 cm long is heated with one end held at temperature 10º C and the other end is insulated. Therefore the boundary conditions for the problem are:

a. $u(0, t) = 10$; $u_x(100, t) = 0$

b. $u(0, t) = 10$; $u(100, t) = 0$

c. $u_x(0, t) = 0$; $u_x(100, t) = 0$

d. $u(x, t) = 0$; $u(100, t) = 0$

e. $u_x(0,t) = 10$; $u_x(100, t) = 0$

(partial credit, next page)

PARTIAL CREDIT (18 points each)

20. A bar for which $\alpha^2 = 3$ is 50 cm long, has both ends held at 10° C. The bar is heated initially in such a way that $u(x, 0) = 10 + 4 \sin \frac{3\pi x}{50}$. Solve the heat equation for $u(x, t)$.

21. Let $f(x) = \sin x$ for $0 \leq x \leq \pi$. Find the cosine series with period 2π for $f(x)$.