

I (8 points).

Which of the sketches on the last page best describe the trajectories of the system

$x' = Ax$. Also determine stability of the system if:

1. $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

Type_____

Stability_____

2. $A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

Type_____

Stability_____

3. $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

Type_____

Stability_____

4. The equation $\frac{d^2 y}{dt^2} - \frac{3dy}{dt} - e^t y = 0$ is equivalent to the first order system

(a) $\frac{dx_1}{dt} = x_1$

(b) $\frac{dx_1}{dt} = x_2$

(c) $\frac{dx_1}{dt} = x_2$

$\frac{dx_2}{dt} = 3x_1 + e^t x_2$

$\frac{dx_2}{dt} = 3x_1 + e^t x_2$

$\frac{dx_2}{dt} = e^t x_1 + 3x_2$

(d) $\frac{dx_1}{dt} = x_2$

(e) $\frac{dx_1}{dt} = -x_2$

$\frac{dx_2}{dt} = x_1 - x_2$

$\frac{dx_2}{dt} = (e^t + 3t)x_2$

5. The critical points of the system $x' = x - xy$ are

$$y' = y(x+2)$$

(a) only (0,0)

(b) (1,2) and (-2,1)

(c) only (-2,1)

(d) (0,0) and (1,2)

(e) (0,0) and (-2,1)

6. A fundamental matrix $\Psi(t)$ for the system $X' = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} X$ is

(a) $\begin{pmatrix} e^{-t} & e^{2t} \\ 0 & 3e^{2t} \end{pmatrix}$

(b) $\begin{pmatrix} e^{2t} & e^{-t} \\ 0 & 3e^{-t} \end{pmatrix}$

(c) $\begin{pmatrix} e^{2t} & e^{-t} \\ 0 & -e^{-t} \end{pmatrix}$

$$(d) \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \quad (e) \begin{pmatrix} e^{-t} & e^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

7. The matrix $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$ has eigenvalues $r = -1 \pm 2i$.

A real solution of the system $X' = AX$ is

$$(a) e \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix} \quad (b) e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} \quad (c) e^{-t} \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$

$$(d) e^{-2t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad (e) e^{-2t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

8. One solution of the system $X' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} X$ is $X^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$. A second solution $X^{(2)}$ is

$$(a) \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t \quad (b) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} t e^t$$

$$(c) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^t \quad (d) \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^t$$

$$(e) \begin{pmatrix} 1 \\ 0 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

II Partial Credit

9. Given the almost linear system

$$\begin{aligned} x' &= y - 1 - x + \sin x \\ y' &= y - x - \cos x \end{aligned}$$

(a) (2 points) Verify that $x=1, y=0$ is a critical point.

(b) (6 points) Find the linear system corresponding to the given system near $(1,0)$.

(c) (6 points) Determine the type and stability of the critical point $(0,1)$.

10. Two linearly independent solutions of the system

$$X' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} X \quad \text{are}$$

$$X^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \quad \text{and} \quad X^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$

(a) (5 points) A fundamental matrix Ψ for this system such that $X(t) = \Psi(t) C$ is

$$\Psi(t) = \underline{\hspace{2cm}}$$

(b) (10 points) Setting $X_p = \Psi(t) U$ and using the method of variation of parameters for the non-homogeneous system

$$X' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} x + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix},$$

find the equation satisfied by the components u'_1 u'_2 , of U .

(c) (7 points) Using the results of part (b) find a particular solution X_p for the non-homogeneous system.