Math 325: Differential Equations Name:__________________________

Exam II *Oct. 31, 1996* Section:__________________________

There are 8 problems worth total of 85 points. You start with 15 points.

1. (10 points)

   a) Use the Laplace transform to solve the initial value problem

   $$y'' - 2y' + 10y = \delta(t), \quad y(0) = y'(0) = 0$$

   where $\delta(t)$ is the Dirac delta function.

   b) Let $f(t)$ be an unspecified function that has a Laplace transform. Express the solution to the following initial value problem as an integral.

   $$y'' - 2y' + 10y = f(t), \quad y(0) = y'(0) = 0$$

2. (15 points) Solve the following initial value problem using Laplace transforms.

$$y'' + 3y' + 2y = g(t), \quad y(0) = 0, y'(0) = 1$$

where

$$g(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases}$$

3. (10 points) Let $\mathbf{X}^{(1)}(t) = \begin{bmatrix} t \\ t \\ t^2 \end{bmatrix}$, $\mathbf{X}^{(2)}(t) = \begin{bmatrix} t^2 \\ t \\ t \end{bmatrix}$, $\mathbf{X}^{(3)}(t) = \begin{bmatrix} t \\ t^2 \\ t \end{bmatrix}$,

a) Compute the Wronskian of $\mathbf{X}^{(1)}(t)$, $\mathbf{X}^{(2)}(t)$, $\mathbf{X}^{(3)}(t)$.

b) For what values of t are the vectors $\mathbf{X}^{(1)}(t)$, $\mathbf{X}^{(2)}(t)$, $\mathbf{X}^{(3)}(t)$ linearly dependent.

c) On what intervals are the vector functions $\mathbf{X}^{(1)}(t)$, $\mathbf{X}^{(2)}(t)$, $\mathbf{X}^{(3)}(t)$ linearly independent.

4. (10 points) Classify the following systems of first order equations using as many of the following terms as apply: linear, non-linear, homogeneous, non-homogeneous. (Do not try to solve the equations!)

a)

$$\begin{aligned} x_1' &= t x_1 + (t^2 - e^t) x_2 \\ x_2' &= -(t^3 - e^{-t}) x_1 + \frac{1}{t} x_2 \end{aligned}$$

b)

$$\begin{aligned} x_1' &= \cos(t) x_2 - \sin(t) \\ x_2' &= \sin(t) x_1 + \cos(t) \end{aligned}$$

c)

$$\begin{aligned} x_1' &= 1/x_2 \\ x_2' &= 1/x_1 \end{aligned}$$

5. (10 points) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 2 & -3 \\ 2 & 4 & 10 \end{bmatrix}$$

6. (10 points) Find the general solution of the system

$$\mathbf{X}' = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 3 & 5 \end{bmatrix} \mathbf{X}$$

7. (10 points) Find the solution of the initial value problem

$$\mathbf{X}' = \begin{bmatrix} 4 & -7 \\ 7 & 4 \end{bmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The coefficient matrix has a complex eigenvlaue of $4 + 7i$ and a corresponding complex eigenvector $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

8. (10 points) Find the general solution of the system

$$\mathbf{X}' = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix} \mathbf{X}$$