Math 325: Differential Equations Name:__________________________

Exam III *Dec. 3, 1996* Section:__________________________

There are 8 problems worth total of 85 points. You start with 15 points.

1. (12 points) Find the fundamental matrix $\Phi(t) = \exp(\mathbf{A}t)$, with $\Phi(0) = \mathbf{I}$, for the homogeneous linear system $\mathbf{X}' = \mathbf{A} \cdot \mathbf{X}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

2. (12 points) Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The general solution to the homogeneous system $\mathbf{X}' = \mathbf{A} \cdot \mathbf{X}$ is

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

Using the method of variation of parameters, find a particular solution of the nonhomogeneous system

$$\mathbf{X}' = \mathbf{A} \cdot \mathbf{X} + \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}$$

3. (12 points) Consider the nonlinear system:

$$\begin{aligned}\frac{dx}{dt} &= x + 2\sin(x) + 2y \\ \frac{dy}{dt} &= 2x + 2y + e^y - 1\end{aligned}$$

a) Find the coefficient matrix for the corresponding linear system at the critical point $(0, 0)$.

b) Find the eigenvalues and eigenvectors for this linear system.

c) Classify the type and stability of the critical point $(0, 0)$ for the nonlinear system.

d) Sketch several trajectories in the phase plane around $(0, 0)$ for the nonlinear system.

4. (5 points) Determine which of the following systems are autonomous.

a)
$$\begin{aligned} \frac{dx}{dt} &= \sin(y) + 3t \\ \frac{dy}{dt} &= x + e^y \end{aligned}$$

b)
$$\begin{aligned} \frac{dx}{dt} &= e^y + 100x \\ \frac{dy}{dt} &= \tan(x) \end{aligned}$$

c)
$$\begin{aligned} \frac{dx}{dt} &= 100x + 7y + e^t \\ \frac{dy}{dt} &= -3x + 9y + \sin(t) \end{aligned}$$

d)
$$\begin{aligned} \frac{dx}{dt} &= y + t \\ \frac{dy}{dt} &= 2 - x \end{aligned}$$

e)
$$\begin{aligned} \frac{dx}{dt} &= x^2 + y^2 \\ \frac{dy}{dt} &= 1 - xy \end{aligned}$$

5. (12 points) Determine all real critical points of the following nonlinear system and discuss their type and stability.

$$\begin{aligned} \frac{dx}{dt} &= x - y \\ \frac{dy}{dt} &= x - y^3 \end{aligned}$$

6. (10 points) The following nonlinear system models populations of competing species:

$$\begin{aligned}\frac{dx}{dt} &= x(2 - y - x)\\ \frac{dy}{dt} &= y(3 - 2x - y)\end{aligned}$$

Determine which phase portrait corresponds to this system. Justify your answer; simply choosing a phase portrait is not sufficient.

(a) (b)

(c) (d)

(e)

7. (12 points) The following nonlinear system models predator-prey populations with logistic constraints that limit the growth of the prey population, x, due to environmental factors:

$$\begin{aligned} \frac{dx}{dt} &= x(3 - x - y) \\ \frac{dy}{dt} &= y(-2 + x) \end{aligned}$$

Determine the limiting populations $x(t)$, $y(t)$ as $t \to \infty$ if the starting populations are $x(0) = 1$ and $y(0) = 3$. Be sure to justify your answer.

8. (10 points) Use the method of separation of variables to replace the partial differential equation

$$(x^2 u_x)_x - u_{tt} = 0$$

by a pair of ordinary differential equations.