Math 325: Differential Equations
Final Exam *Dec. 19, 1996*

Name:________________________
Section:_______________________

There are 15 problems worth total of 150 points. Be sure to justify your answers and show all your work.

1. (10 points) Find the general solution of the nonhomogeneous differential equation

$$y''' + y = xe^{-x}$$

2. (10 points) Use the Runge-Kutta method with step size $h = 0.1$ to find an approximate value of the solution at $t = 1.1$ of the initial value problem

$$y' = t + e^y, \qquad y(1) = 0$$

Be sure to indicate carefully the formulas you are using and show every step of the calculation to four decimal places. (If you cannot remember the Runge-Kutta formulas, you can take a 3 point penalty and solve this problem using the Euler method.)

3. (10 points)

a) Define the Laplace transform $\mathcal{L}\{f(t)\}$ of a function $f(t)$.

b) Write $\mathcal{L}\{f^{(n)}(t)\}$ in terms of $\mathcal{L}\{f(t)\}$ and the initial values of $f(t)$ and its derivatives.

4. (10 points) Use the Laplace transform to solve the initial value problem

$$y'' + y = \delta(t - \pi) - u_{2\pi}(t), \qquad y(0) = 1, \quad y'(0) = 0$$

where $\delta(t)$ is the unit impulse function and $u_{2\pi}(t)$ is a unit step function.

5. (10 points)

a) Define the convolution $f * g(t)$ of two functions $f(t)$ and $g(t)$.

b) Express $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4(s^2+1)}\right\}$ as an integral.

6. (10 points) Prove that the functions e^x, e^{-x}, e^{2x} are linearly independent.

7. (10 points) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}$$

8. (10 points) Find the general solution of the system $\mathbf{X}' = \mathbf{A} \cdot \mathbf{X}$ where $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix}$$

9. (10 points) Use the method of variation of parameters to find a particular solution of the system

$$\mathbf{X}' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \mathbf{X} + \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

10. (10 points) Find all critical points of the system

$$\begin{aligned} \frac{dx}{dt} &= y^2 - 1 \\ \frac{dy}{dt} &= x^3 - y \end{aligned}$$

and determine their type and stability.

11. (10 points) Consider the predator-prey system

$$\begin{aligned} \frac{dx}{dt} &= x - xy \\ \frac{dy}{dt} &= -y + xy \end{aligned}$$

a) Find the critical points of this system and describe their stability properties.

b) Sketch the phase portrait for this system.

12. (10 points) Solve the heat equation $3u_{xx} = u_t$ subject to the homogeneous boundary conditions

$$u(0,t) = 0 = u(50,t), \qquad t > 0$$

and initial condition

$$u(x,0) = 100$$

13. (10 points) Solve the heat equation $\alpha^2 u_{xx} = u_t$ subject to the non-homogeneous boundary conditions

$$u(0,t) = 0, \quad u(\ell, t) = 50, \qquad t > 0$$

and initial condition

$$u(x,0) = \frac{50}{\ell} x$$

14. (10 points) Solve the wave equation $4u_{xx} = u_{tt}$ subject to the homogeneous boundary conditions

$$u(0,t) = 0 = u(\pi,t)$$

and initial conditions

$$\begin{aligned} u(x,0) &= \frac{1}{10}\sin(2x) \\ u_t(x,0) &= 0 \end{aligned}$$

15. (10 points) Find the Fourier series expansion for the function

$$f(x) = \begin{cases} x+2 & -2 \le x \le 0 \\ 2-x & 0 \le x \le 2 \end{cases}$$