Name: ______________________ **Instructor:** ______________

Mathematics 325
Fall Semester 2000
Final Exam.
December 13, 2000

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1. This Examination contains 21 problems worth a total of 150 points. The test will be exactly 120 minutes in length.

2. For each multiple choice question, please mark an X on the correct answer on the answer sheet. **Do not circle it.**

3. Calculators, books and notes are not allowed.

4. A table of Laplace transforms is included at the end of the booklet.

5. Hand in the entire test.

Sign the pledge: "On my honor, I have neither given nor received unauthorized aid on this Exam":

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HAVE A NICE VACATION

1. The Wronskian of $\{1, e^{2x}, e^{-2x}\}$ is
   (a) 16
   (b) 2
   (c) e^{2x}
   (d) e^{4x}
   (e) 1

2. The general solution of the equation

$$y^{(3)} + 3y'' + 2y' = 0$$

is
   (a) $c_1 + c_2e^{-t} + c_3e^{-2t}$
   (b) $c_1e^{3t} + c_2e^{2t} + c_3e^{t}$
   (c) $c_1 + c_2e^{t} + c_3e^{2t}$
   (d) $c_1e^{-t} + c_2e^{-2t} + c_3e^{-3t}$
   (e) $c_1 + c_2t + c_3t^2$

3. Determine a suitable form for a particular solution of the equation

$$y''' - 9y' = t + e^{3t}.$$

   (a) $t(A_0t + A_1) + Bte^{3t}$
   (b) $A_0t + A_1 + Be^{3t}$
   (c) $t(A_0t + A_1) + Be^{3t}$
   (d) $t(A_0t + A_1) + B$
   (e) $A_0t + Bte^{3t}$

4. Given that $y_1(t) = 1$, $y_2(t) = t$ and $y_3(t) = e^t$ form a fundamental set of solutions of the equation $y''' - y'' = 0$, determine a particular solution of

$$y''' - y'' = g(t)$$

in terms of an integral.
   (a) $\int_{t_0}^{t}(s-1)g(s)ds - t\int_{t_0}^{t} g(s)ds + e^t\int_{t_0}^{t} e^{-s}g(s)ds$
   (b) $\int_{t_0}^{t} e^s(s-1)g(s)ds - t\int_{t_0}^{t} e^s g(s)ds + e^t\int_{t_0}^{t} g(s)ds$
   (c) $\int_{t_0}^{t} e^s(s-1)g(s)ds - \int_{t_0}^{t} e^s g(s)ds + \int_{t_0}^{t} g(s)ds$
   (d) $\int_{t_0}^{t}(s-1)g(s)ds - \int_{t_0}^{t} g(s)ds + \int_{t_0}^{t} e^{-s}g(s)ds$
   (e) $\int_{t_0}^{t} sg(s)ds - t\int_{t_0}^{t} e^s g(s)ds + e^t\int_{t_0}^{t} e^{-s}g(s)ds$

5. Let

$$f(t) = \begin{cases} 100, & 0 < t \le 99 \\ 0, & t > 99. \end{cases}$$

Find the Laplace transform of f.

(a) $\dfrac{100}{s}(1 - e^{-99s})$

(b) $\dfrac{99}{s}(1 - e^{-100s})$

(c) $100(1 - e^{-99s})$

(d) $99(1 - e^{-100s})$

(e) $\dfrac{100}{s}(1 - e^{99s})$

6. The inverse Laplace transform of

$$\frac{2s+5}{s^2+5s+6}$$

is

(a) $e^{-2t} + e^{-3t}$

(b) $e^{2t} + e^{3t}$

(c) $\sin 2t + \sin 3t$

(d) $e^{2t} \cos 2t + e^{3t} \sin 3t$

(e) $e^{3t} \cos 2t$

7. The solution of the initial value problem

$$y'' + 4y = \delta(t - \frac{\pi}{4}), \quad y(0) = y'(0) = 0$$

is

(a) $\frac{1}{2} \sin 2(t - \frac{\pi}{4})$

(b) $\frac{1}{2} \cos 2(t - \frac{\pi}{4})$

(c) $\delta(t - \frac{\pi}{4}) \sin 2(t - \frac{\pi}{4})$

(d) $\frac{1}{2} u_{\frac{\pi}{4}}(t) \sin 2(t - \frac{\pi}{4})$

(e) $\frac{1}{2} u_{\frac{\pi}{4}}(t) \cos 2(t - \frac{\pi}{4})$

8. Find the general solution of the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$$

(a) $c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 (te^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$

(b) $c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) $c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 te^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(d) $c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 (te^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix})$

(e) $c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

9. Let us consider the first order system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

Suppose that the matrix A has a complex eigenvalue $r = 1 + i$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is the corresponding eigenvector. Determine which of the following is the general solution.

(a) $c_1 e^t (\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix}) + c_2 e^t (\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix})$

(b) $c_1 e^t (\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix}) + c_2 e^t (\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix})$

(c) $c_1 e^t (\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix}) + c_2 e^t (\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix})$

(d) $c_1 e^t \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

(e) $c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

10. Find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ for the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \mathbf{x}.$$

(a) $\frac{1}{2} \begin{pmatrix} e^{2t} + e^{-2t}, & e^{2t} - e^{-2t} \\ e^{2t} - e^{-2t}, & e^{2t} + e^{-2t} \end{pmatrix}$

(b) $\begin{pmatrix} e^{2t}, & e^{-2t} \\ e^{2t}, & -e^{-2t} \end{pmatrix}$

(c) $\begin{pmatrix} e^{-2t}, & e^{2t} \\ e^{-2t}, & -e^{2t} \end{pmatrix}$

(d) $\frac{1}{2} \begin{pmatrix} e^{2t} + e^{-2t}, & -e^{2t} + e^{-2t} \\ e^{2t} - e^{-2t}, & e^{2t} + e^{-2t} \end{pmatrix}$

(e) $\frac{1}{2} \begin{pmatrix} e^{2t} + e^{-2t}, & e^{2t} - e^{-2t} \\ -e^{2t} + e^{-2t}, & e^{2t} + e^{-2t} \end{pmatrix}$

11. Given that a fundamental matrix for the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

is

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix},$$

find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \tan t \\ e^{-t} \end{pmatrix}$$

is

(a) $c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \int \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix} \begin{pmatrix} \tan t \\ e^{-t} \end{pmatrix} dt.$

(b) $c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int \begin{pmatrix} \tan x \\ e^{-t} \end{pmatrix} dt.$

(c) $c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \int \begin{pmatrix} e^{t} & -e^{3t} \\ e^{t} & e^{3t} \end{pmatrix} \begin{pmatrix} \tan t \\ e^{-t} \end{pmatrix} dt.$

(d) $c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \int \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} \tan t \\ e^{-t} \end{pmatrix} dt.$

(e) $c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \int \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} \tan t \\ e^{-t} \end{pmatrix} dt.$

12. Consider the almost linear system

$$\frac{dx}{dt} = x - 2y + 3y^2$$
$$\frac{dy}{dt} = y - 2xy + 3x^2.$$

The critical point $(0, 0)$ is

(a) unstable and its type for the non-linear system cannot be determined from the corresponding linear system.

(b) an unstable improper node for both non-linear system and the corresponding linear system.

(c) unstable for the corresponding linear system and stable for the non-linear system.

(d) The stability for the non-linear system cannot be determined from the corresponding linear system.

(e) an unstable improper node for the corresponding linear system but an unstable node for the non-linear system.

13. For the almost linear system

$$\frac{dx}{dt} = 2y - xy - y^2$$
$$\frac{dy}{dt} = -x + 3x^2 - 2xy,$$

(a) $(1, 1)$ is an asymptotically stable spiral point.

(b) $(1, 1)$ is either a spiral point or a center. Its stability can not be determined from the corresponding linear system.

(c) $(1, 1)$ is an unstable saddle point.

(d) $(1, 1)$ is an asymptotically stable node.

(e) The type of the critical point $(1, 1)$ cannot be determined from the corresponding linear system.

14. The system

$$\frac{dx}{dt} = -x^3 + xy^2$$
$$\frac{dy}{dt} = -3xy^2$$

has $(0,0)$ as a critical point. Using $V(x,y) = 3x^2 + y^2$ as a Liapunov function to test the stability of $(0,0)$, what conclusion can we get?

(a) $(0,0)$ is a stable critical point.

(b) $(0,0)$ is a saddle point.

(c) $(0,0)$ is an unstable critical point.

(d) No conclusion can be drawn by using this Liapunov function. We should use another Liapunov function to test the stability of $(0,0)$.

(e) The stability of $(0,0)$ cannot be determined by Liapunov's method.

15. Suppose $y' = (y+t)^2$, $y(2) = 1$. Approximate $y(2.2)$ using Euler's Method, with $h = 0.1$.
(a) 3.5
(b) 3.2
(c) 3.3
(d) 3.4
(e) 3.6

16. If $u_{xx}(x,t) = 2u_{xt}(x,t)$, where $u(x,t) = X(x)T(t)$, which pair of equations is satisfied separately by $X(x)$ and $T(t)$?
(a) $X'' + \lambda X' = 0$, $2T' + \lambda T = 0$
(b) $X'' + \lambda X = 0$, $T'' + 2\lambda T = 0$
(c) $X' + \lambda X = 0$, $T' + \lambda T = 0$
(d) $X'' + \lambda X' + X = 0$, $T = 2\lambda$
(e) there is no solution of form $u(x,t) = X(x)T(t)$

17. Suppose that

$$f(x) = \begin{cases} 0, & -2 \le x < 0 \\ 1, & 0 \le x < 2. \end{cases}$$

and $f(x+4) = f(x)$. If $a_0 + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{2}x) + b_n \sin(\frac{n\pi}{2}x))$ is the Fourier series for $f(x)$, what is b_3?
(a) $\frac{2}{3\pi}$
(b) $\frac{3\pi}{2}$
(c) $\frac{1}{6\pi}$
(d) 0
(e) $\frac{3\pi}{4}$

18. Which is the value does the Fourier series from Problem 4 at $x = 2$?
(a) $\frac{1}{2}$
(b) 1
(c) 0
(d) $\frac{1}{4}$
(e) the series does not converge at $x = 2$

19. If $f(x) = x$, $0 \leq x < 2$, and $f(x+4) = f(x)$, what is the value of the even extension of $f(x)$ at $x = 3$?
(a) 1
(b) 0
(c) 2
(d) 3
(e) -1

20. A bar of metal is located on the x-axis between $x = 0$ and $x = 2$. The bar is insulated, and both ends are maintained at temperature 0. At time 0, the temperature at position x is given by $f(x) = \sin(\frac{\pi}{2}x) + 2\sin(\pi x)$, for $0 < x < 2$. Assume $\alpha = 2$, so that $u_{xx}(x,t) = 4u_t(x,t)$. Find the temperature $u(x,t)$ for $0 < x < 2$ and $t > 0$.
(a) $\sin(\frac{\pi}{2}x)e^{-\pi^2 t} + 2\sin(\pi x)e^{-4\pi^2 t}$
(b) $\sin(\frac{\pi}{2}x)\cos(\pi t) + 2\sin(\pi x)\cos(2\pi t)$
(c) $\sin(\frac{2}{\pi}x)\cos(\frac{1}{\pi}t) + 2\sin(\frac{1}{\pi}x)\cos(\frac{1}{2\pi}t)$
(d) $\sin(\frac{2}{\pi}x)e^{-\pi t} + 2\sin(\frac{1}{\pi}x)e^{-2\pi t}$
(e) $u(x,t) = 0$

21. Suppose $f(x)$ is piecewise continuous on the interval $[0,3]$, with Fourier sine series $\sum_{n=1}^{\infty} \frac{1}{n^2}\sin(\frac{n\pi}{3}x)$. Find the solution for the wave equation $u_{xx}(x,t) = u_{tt}$, boundary conditions $u(0,t) = 0$, $u(3,0) = 0$, for $t \geq 0$, and initial conditions $u_t(x,0) = 0$ and $u(x,0) = f(x)$.
(a) $u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(\frac{n\pi}{3}x)\cos(\frac{n\pi}{3}t)$
(b) $u(x,t) = \frac{2\pi}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}(\sin(n\pi x)\cos(\frac{n\pi}{3}t)$
(c) $u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin(\frac{n^2\pi^2}{9}x)\cos(\frac{n^2\pi^2}{9}t)$
(d) $u(x,t) = \frac{9}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin(\frac{n\pi}{3}x)e^{-\frac{n^2\pi^2}{9}t}$
(e) $u(x,t) = 0$