Maple Output Maple Plot Warning 2D Math 2D Output

Math 325: Differential Equations Name:________________________

Final *May 9, 2000*

A table of Laplace transforms and a table of Fourier series are attached. You may use your own calculator. You may also use a summary (two sides of an $8\frac{1}{2}'' \times 11''$ sheet of paper, with notes in your writing). You may not use anything else. You may not pass a calculator or summary to another person.

Show all your work. Erase or cross out any work you do not want graded. There are 6 questions on 9 pages plus the bonus question page and two pages of tables.

1. (20 points) Solve

$$y'' + 4y = 2\delta(t - \frac{\pi}{4}), \qquad y(0) = 0,\ y'(0) = 0.$$

2. (20 points) Let

$$A = \frac{1}{65}\begin{bmatrix} 190 & 10 & 120 & 35 & -10 \\ 4 & 226 & 86 & 232 & -421 \\ 385 & 10 & -75 & 35 & -10 \\ 521 & 154 & 184 & -72 & -154 \\ 366 & 9 & 134 & 38 & -204 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ -1 & 2 & 1 & -1 & 0 \\ 1 & 1 & 0 & 3 & -1 \\ 1 & 2 & 0 & 1 & 4 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}.$$

Then

$$B^{-1}AB = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

Find the general solution of $\mathbf{x}' = A\mathbf{x}$.

3. (20 points) Solve:

$$\mathbf{x}''(t) + \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}^2 \mathbf{x}(t) = \mathbf{0},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{x}'(0) = \begin{bmatrix} 8 \\ 16 \end{bmatrix}.$$

(**Hint:** Suppose D is a diagonal matrix. How would you solve $\mathbf{y}'' + D^2\mathbf{y} = \mathbf{0}$?)

4. (15 points) Solve:

$$
\begin{aligned}
&u_t = u_{xx}, \qquad 0 < x < \pi,\ t > 0, \\
&u(0,t) = 0 = u(\pi,t), \qquad t > 0, \\
&u(x,0) = \sqrt{321}\sin 3x - 59\sin 7x, \qquad 0 < x < \pi.
\end{aligned}
$$

5. (20 points) The undamped pendulum satisfies the equations

$$x' = y$$
$$y' = -\sin x$$

where $x = \theta$ is the angle the pendulum makes with the downward vertical direction and $y = \theta'$ is the angular velocity.

(a) Find all equilibrium points of the system.

(b) Show that the energy of the pendulum

$$E(x, y) = \frac{1}{2}y^2 + 1 - \cos x$$

is constant on trajectories of the system.

(c) Show that $E(\pi, 0) = 2$ and $E(0, 2) = 2$.

(d) Consider the following snapshot of Chuck's Maple session.
active1dwith(plots):
active1dsys := diff(x(t),t) = y(t), diff(y(t),t)=-sin(x(t)):
active1dnumsol := b -¿ dsolve({sys,x(0)=0,y(0)=b},{x(t),y(t)},numeric):
active1dcurve := (b,range) -¿ odeplot(dsolve({sys,x(0)=0,y(0)=b},{x(t),y(t)} ,numeric),[x(t),y(t)],range,scaling=CONSTRAINED,numpoints=300):
active1dnphase := range -¿ display({seq(curve(0.5*b,range),b=1..5)},view=[-10..10,-5..5]):
active1dnphase(0..20);

pendulum01.eps

What is wrong with the plot? What probably caused Maple to make this mistake?

6. (20 points) Consider the following snapshot of Carol's Maple session.
active1dsolve({-y*(x-3)=0,(x-2)*(x-2*y+1)=0},{x,y});
inert2d{x = 2, y = 0}, {y = 0, x = -1}, {y = 2, x = 3};

$$\{x = 2,\ y = 0\},\ \{y = 0,\ x = -1\},\ \{y = 2,\ x = 3\}$$

active1dwith(DEtools):
active1ddfieldplot([diff(x(t),t)=-y(t)*(x(t)-3),diff(y(t),t)=(x(t)-2)*(x(t)-2 *y(t)+1)],[x(t),y(t)],t=0..1,
2..4,y=-1..3,
active1darrows=SLIM,axes=BOXED,scaling=CONSTRAINED);

p601.eps

active1dwith(linalg):
Warning, new definition for adjoint
Warning, new definition for norm
Warning, new definition for trace
active1dj1x := diff(-y*(x-3),x): j1y := diff(-y*(x-3),y):
active1dj2x := diff((x-2)*(x-2*y+1),x): j2y := diff((x-2)*(x-2*y+1),y):
active1djacobmat := [[j1x,j1y],[j2x,j2y]]:
active1dpoint1 := subs({x=3,y=2},jacobmat):
active1dpoint2 := subs({x=-1,y=0},jacobmat):
active1dpoint3 := subs({x=2,y=0},jacobmat):
active1deigenvals(point1); eigenvals(point2); eigenvals(point3);
inert2d-2, -2;

$$-2,\ -2$$

inert2d3+I*sqrt(3), 3-I*sqrt(3);

$$3 + I\,\sqrt{3},\ 3 - I\,\sqrt{3}$$

inert2dsqrt(3), -sqrt(3);

$$\sqrt{3},\ -\sqrt{3}$$

(a) What system of differential equations is Carol studying?

(b) What are the critical points of the system?

(c) The Maple session contains enough information to identify the type and stability of all but one of the critical points. For each critical point, give its type and stability or explain why the information in the Maple session is inconclusive. Justify your answer based on the Jacobian information found in the Maple session.

(d) If one imposes the initial condition $x(0) = -1,\ y(0) = -0.1$, the limiting behavior of the solution trajectory as $t \to \infty$ is difficult to ascertain. Identify at least two possibilities for the limiting behavior of the solution trajectory.

Bonus question (10 points)
Find:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

TABLE OF FOURIER TRANSFORMS

The functions in the table are defined for $-\pi < x < \pi$.

function	*Fourier series*
x	$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$
$\lvert x \rvert$	$\frac{\pi}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos nx$
x^2	$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$
x^3	$2\sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right) \sin nx$
$f(x)$	$\sum_{n=1}^{\infty} \frac{2}{n\pi} (1-(-1)^n) \sin n\pi$

The function f in the table is equal to 1 if $0 < x < \pi$ and -1 if $-\pi < x < 0$.