Math 325

Take-home Problem for Final Exam, 35 points

Due at 4:15 p.m., Tuesday, May 9, 2000

You may consult your course notes, homework, and the textbooks. The problem requires using Maple. You may not consult any other books or notes. You may not discuss the exam with anyone except Prof. Stanton.

In this problem you will investigate the effect of a periodic, possibly discontinuous, forcing function on a second order linear equation with constant coefficients. Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0,\ y'(0) = 1. \tag{1}$$

The general solution of the associated homogeneous equation is

$$y(t) = A\cos t + B\sin t$$

which is periodic with period 2π. Recall that the phenomenon of *resonance*, that is, solutions which are unbounded as $t \to \infty$, occurs when the forcing function $h(t)$ is a linear combination of $\cos t$ and $\sin t$. You will investigate whether resonance occurs when the forcing function is periodic of period 2π but not necessarily continuous. The problem is similar to Problem Set E #17 in *Differential Equations with Maple*.

A note about terminology: If you are asked to **explain** an answer, you are being asked to give a mathematical justification, that is, a mathematical proof that the answer is correct.

(a) Using step functions, define a function $H(t)$ a whose value is t on $[0, 2\pi)$, $t - 2\pi$ on $[2\pi, 4\pi)$, $t - 4\pi$ on $[4\pi, 6\pi)$ and so on. Define a Maple function $h(t)$ which agrees with $H(t)$ on the interval $[0, 10\pi]$. Plot the Maple function on the interval $[0, 30]$. It should have the appearance of a sawtooth wave. (Figure 6.3.8 on p. 313 of Boyce and DiPrima shows a sawtooth wave which has period 1.)

(b) Use **dsolve** with **method=laplace** to solve equation (1) with the function $h(t)$ defined in part (a). Explain why the solution will agree on the interval $[0, 30]$ with the one having $H(t)$ forcing function. Plot the solution together with $h(t)$ on the interval $[0, 30]$. Do you see resonance? Compute and plot a numerical solution to confirm your answer.

(c) In part (a), you constructed a forcing function $H(t)$ with period 2π. The function $H(t/2)$ has period 4π. Repeat part (b) using the forcing function $h(t/2)$. Do you see resonance? If necessary, use a larger interval.

(d) Repeat part (b) using the forcing function $h(2t)$. Do you see resonance? What is the period of $H(2t)$?

(e) What can you conclude about the resonance effect for discontinuous forcing functions? Would you expect resonance to occur in equation (1) for *any* forcing function of period 2π? (*Hint*: Is the function $H(2t)$ periodic with period 2π?) When do you expect resonance to occur for piecewise continuous 2π periodic forcing functions? Explain your answer. (*Hint*: Consider the 2π periodic Fourier series of the forcing function.) You might try some other periodic forcing functions to check your answer.

Bonus Question, 10 points

(a) Which computer problem did you find the most valuable? Why?

(b) Which computer demonstration did you like the most? Why?

(**Note:** Full credit will be given for a thoughtful answer on this question.)