Math 325
Spring, 2000

GROUP PROJECT 1, due Wednesday, April 5.

Let

$$\begin{aligned} x' &= f(x, y, \mu) \\ y' &= g(x, y, \mu) \end{aligned} \tag{1}$$

be a planar autonomous system of ordinary differential equations depending on a parameter μ. The purpose of this project is to study the dependence of the behavior of the system on the parameter μ. This is a group project. Please follow the guidelines for group projects.

Assume that f and g vanish at the origin. We can write the system (1) in the form

$$\mathbf{x}' = A(\mu)\mathbf{x} + Q(\mathbf{x}, \mu)$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix},$$

$A(\mu)$ is a matrix depending on μ but not on $\mathbf{x}$, and Q contains the higher order terms in (1). The matrix $A(\mu)$ is called the *linearization* of (1) at the origin.

I. Consider the system

$$\begin{aligned} x' &= \mu x + y - x(x^2 + y^2) \\ y' &= -x + \mu y - y(x^2 + y^2). \end{aligned} \tag{2}$$

1. Show that the origin is the only equilibrium point of the system (2).

2. Find the eigenvalues of the linearization of (2). Show that they are of the form $\alpha(\mu) + i\beta(\mu)$ where α and β are differentiable functions of μ, $\alpha(0) = 0$, $\alpha'(0) > 0$ and $\beta(0) \neq 0$.

3. Identify the type of critical point and stability of the linearization. (The answer will depend on μ.)

4. For $\mu = -1, -0.5, 0, 0.5, 1$ plot trajectories of the linearization and describe the behavior of the trajectories as $t \to \infty$.

5. For the same values of μ, plot trajectories of the system (2). What do you notice about the behavior of the trajectories as $t \to \infty$? How does it depend on μ?

6. Rewrite the system (2) in polar coordinates. You should get the system

$$\begin{aligned} r' &= r(\mu - r^2) \\ \theta' &= -1. \end{aligned} \tag{3}$$

7. Use the equations (3) to analyze the behavior of trajectories of (2) as $t \to \infty$.

a. Show that if $\mu \leq 0$, the trajectories spiral clockwise toward the origin as $t \to \infty$.

b. Show that if $\mu > 0$ the circle $r = \sqrt{\mu}$ is a trajectory.

c. Show that if $\mu > 0$ then a trajectory through a point with $0 < r < \sqrt{\mu}$ spirals clockwise outward to $r = \sqrt{\mu}$ as $t \to \infty$ and a trajectory through a point with $\sqrt{\mu} < r$ spirals clockwise inward to $r = \sqrt{\mu}$ as $t \to \infty$. Such a limiting trajectory is called a *limit cycle.*

Hints: You should have seen the behavior described in a-c in your plots. For a and c, use the information you obtain from the sign of the derivatives in (3).

This is an example of a *bifurcation* of an asymptotically stable equilibrium point into a limit cycle. This kind of behavior is called a *Hopf bifurcation.*

In the next three groups of problems, you will explore a linear system and two nonlinear systems with the same linearization. In the final problem you will compare the three systems.

II. Consider the system

$$\begin{aligned} x' &= \mu x - y \\ y' &= x. \end{aligned} \tag{4}$$

8. Show that the origin is the only equilibrium point of (4).

9. Find the eigenvalues of the linearization of (4) and show that they have the same properties as the eigenvalues in problem 2 if $|\mu|$ is small.

10. Identify the type of critical point and stability of (4).

11. For $\mu = -3, -2, -1, -0.5, 0, 0.5, 1, 2, 3$ plot trajectories of (4) and describe the behavior of the trajectories as $t \to \infty$.

12. Show that for $\mu = 0$, the trajectories, other than the equilibrium point, are circles centered at the origin.

III. Consider the system

$$\begin{aligned} x' &= \mu x - y - x^3 \\ y' &= x. \end{aligned} \tag{5}$$

13. Show that the origin is the only equilibrium point of (5).

14. For the values of μ in 11, plot trajectories of the system (5). What do you notice about the behavior of the trajectories as $t \to \infty$? How does it depend on μ?

15. For $0 < \mu < 2$, (5) has a limit cycle. Label its approximate location in your plots in 13. Estimate the x coordinate of the intersection of the limit cycle with the positive x axis as accurately as you can. Explain how you got your estimate.

16. Write (5) in polar coordinates and show that for $0 = \mu$, r is a decreasing function of t on trajectories of the system. From this, how do you expect trajectories to behave as $t \to \infty$? Is this expectation consistent with the behavior you found in your plot for $\mu = 0$?

The system (5), known as the *van der Pol* system, also has a Hopf bifurcation at $\mu = 0$.

IV. Consider the system

$$\begin{aligned} x' &= \mu x - y - y^3 \\ y' &= x. \end{aligned} \tag{6}$$

17. Show that the origin is the only equilibrium point of (6).

18. For the values of μ in 11, plot trajectories of the system (6). What do you notice about the behavior of the trajectories as $t \to \infty$? How does it depend on μ? Did you find limit cycles for $0 < \mu < 2$?

19. Show that the energy function

$$H(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{y^4}{4}$$

is constant on trajectories of (6) for $\mu = 0$. Here $\frac{x^2}{2}$ is the potential energy and $\frac{y^2}{2} + \frac{y^4}{4}$ is the kinetic energy of the system. Conclude that the trajectories are closed curves. You should have observed these in your plot.

The system (6) exhibits a different type of bifurcation at $\mu = 0$.

V. 20. Compare the behavior of the trajectories of (4), (5) and (6) as $t \to \infty$ for the values of μ in problem 11.