Math 325
Spring, 2001

Jordan Canonical Form

A matrix B is a **Jordan block** if it is either of the form

$$B = \lambda I_{1\times 1} \tag{1}$$

where $I_{1\times 1}$ is the 1×1 identity matrix or of the form

$$B = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix} \tag{2}$$

Notice that a matrix of the form (**??**) has zeros below the diagonal, the same number λ in each entry on the diagonal, a 1 in each entry just above the diagonal and zeros every place else above the diagonal.

Theorem 1 *If A is a complex $n \times n$ matrix, there is an invertible matrix U such that*

$$U^{-1}AU = J = \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{bmatrix} \tag{3}$$

where each matrix B_i, $i = 1 \dots k$, is a Jordan block.

The matrix J is called the **Jordan canonical form** of A. The entries on the diagonal of J are the eigenvalues of A. If A is diagonalizable, J is a diagonal matrix, which we usually call D. So, the interest of this theorem is that it gives a similarity transformation from A to a matrix of a simple form for non-diagonalizable matrices A.

A **generalized eigenvector** of an $n \times n$ matrix A corresponding to the eigenvalue λ is a nonzero vector η which solves the equation

$$(A - \lambda I)^k \eta = 0. \tag{4}$$

For $k = 1$ a solution η_1 to (**??**) is just an eigenvector. To obtain generalized eigenvectors for $k \geq 2$, let η_2 solve

$$(A - \lambda I)\eta_2 = \eta_1, \qquad (A - \lambda I)\eta_3 = \eta_2, \tag{5}$$

etc.

How do you find the matrix U? If the eigenvalues λ_i of the Jordan blocks B_i are distinct, you can let U_i be the matrix whose jth column is $\eta_j^{(i)}$ where $\eta_j^{(i)}$ is the jth generalized eigenvector of A with eigenvalue λ_i,

$$(A - \lambda_i I)^j \eta_j^{(i)} = 0$$

but

$$(A - \lambda_i I)^{j-1} \eta_j^{(i)} \neq 0.$$

Then

$$U = [\ U_1 \quad U_2 \quad \ldots \quad U_k\ ]. \tag{6}$$

In general, the columns of U will be generalized eigenvectors of A, but the ones corresponding to two different Jordan blocks with the same eigenvalue will be harder to find.