

Math 325
Spring, 2001

GROUP PROJECT 1, due Wednesday, April 4

Let

$$\begin{aligned}x' &= f(x, y, \mu) \\y' &= g(x, y, \mu)\end{aligned}\tag{1}$$

be a planar autonomous system of ordinary differential equations depending on a parameter μ . The purpose of this project is to study the dependence of the behavior of the system on the parameter μ . This is a group project. Please follow the guidelines for group projects.

Assume that f and g vanish at the origin. We can write the system (??) in the form

$$\mathbf{x}' = A(\mu)\mathbf{x} + Q(\mathbf{x}, \mu)$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix},$$

$A(\mu)$ is a matrix depending on μ but not on \mathbf{x} , and Q contains the higher order terms in (??). The matrix $A(\mu)$ is called the *linearization* of (??) at the origin.

I. Consider the system

$$\begin{aligned}x' &= \mu x + y - x(x^2 + y^2) \\y' &= -x + \mu y - y(x^2 + y^2).\end{aligned}\tag{2}$$

1. Show that the origin is the only equilibrium point of the system (??).
2. Find the eigenvalues of the linearization of (??). Show that they are of the form $\alpha(\mu) + i\beta(\mu)$ where α and β are differentiable functions of μ , $\alpha(0) = 0$, $\alpha'(0) > 0$ and $\beta(0) \neq 0$.
3. Identify the type of critical point and stability of the linearization. (The answer will depend on μ .)
4. For $\mu = -1, -0.5, 0, 0.5, 1$ plot trajectories of the linearization and describe the behavior of the trajectories as $t \rightarrow \infty$.
5. For the same values of μ , plot trajectories of the system (??). What do you notice about the behavior of the trajectories as $t \rightarrow \infty$? How does it depend on μ ?
6. Rewrite the system (??) in polar coordinates. You should get the system

$$\begin{aligned}r' &= r(\mu - r^2) \\ \theta' &= -1.\end{aligned}\tag{3}$$

7. Use the equations (??) to analyze the behavior of trajectories of (??) as $t \rightarrow \infty$.
- Show that if $\mu \leq 0$, the trajectories spiral clockwise toward the origin as $t \rightarrow \infty$.
 - Show that if $\mu > 0$ the circle $r = \sqrt{\mu}$ is a trajectory.
 - Show that if $\mu > 0$ then a trajectory through a point with $0 < r < \sqrt{\mu}$ spirals clockwise outward to $r = \sqrt{\mu}$ as $t \rightarrow \infty$ and a trajectory through a point with $\sqrt{\mu} < r$ spirals clockwise inward to $r = \sqrt{\mu}$ as $t \rightarrow \infty$. Such a limiting trajectory is called a *limit cycle*.

Hints: You should have seen the behavior described in a-c in your plots. For a and c, use the information you obtain from the sign of the derivatives in (??).

This is an example of a *bifurcation* of an asymptotically stable equilibrium point into a limit cycle. This kind of behavior is called a *Hopf bifurcation*.

In the next three groups of problems, you will explore a linear system and two nonlinear systems with the same linearization. In the final problem you will compare the three systems.

II. Consider the system

$$\begin{aligned}x' &= \mu x - y \\y' &= x.\end{aligned}\tag{4}$$

- Show that the origin is the only equilibrium point of (??).
- Find the eigenvalues of the linearization of (??) and show that they have the same properties as the eigenvalues in problem 2 if $|\mu|$ is small.
- Identify the type of critical point and stability of (??).
- For $\mu = -3, -2, -1, -0.5, 0, 0.5, 1, 2, 3$ plot trajectories of (??) and describe the behavior of the trajectories as $t \rightarrow \infty$.
- Show that for $\mu = 0$, the trajectories, other than the equilibrium point, are circles centered at the origin.

III. Consider the system

$$\begin{aligned}x' &= \mu x - y - x^3 \\y' &= x.\end{aligned}\tag{5}$$

- Show that the origin is the only equilibrium point of (??).
- For the values of μ in 11, plot trajectories of the system (??). What do you notice about the behavior of the trajectories as $t \rightarrow \infty$? How does it depend on μ ?
- For $0 < \mu < 2$, (??) has a limit cycle. Label its approximate location in your plots in 14. Estimate the x coordinate of the intersection of the limit cycle with the positive x axis as accurately as you can. Explain how you got your estimate.

16. Write (??) in polar coordinates and show that for $\mu = 0$, r is a decreasing function of t on trajectories of the system. From this, how do you expect trajectories to behave as $t \rightarrow \infty$? Is this expectation consistent with the behavior you found in your plot for $\mu = 0$?

The system (5), known as the *van der Pol* system, also has a Hopf bifurcation at $\mu = 0$.

IV. Consider the system

$$\begin{aligned}x' &= \mu x - y - y^3 \\y' &= x.\end{aligned}\tag{6}$$

17. Show that the origin is the only equilibrium point of (??).

18. For the values of μ in 11, plot trajectories of the system (??). What do you notice about the behavior of the trajectories as $t \rightarrow \infty$? How does it depend on μ ? Did you find limit cycles for $0 < \mu < 2$?

19. Show that the energy function

$$H(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{y^4}{4}$$

is constant on trajectories of (??) for $\mu = 0$. Here $\frac{x^2}{2}$ is the potential energy and $\frac{y^2}{2} + \frac{y^4}{4}$ is the kinetic energy of the system. Conclude that the trajectories are closed curves. You should have observed these in your plot.

The system (??) exhibits a different type of bifurcation at $\mu = 0$.

V. 20. Compare the behavior of the trajectories of (??), (??) and (??) as $t \rightarrow \infty$ for the values of μ in problem 11.