Math 325: Differential Equations Name:________________________

Final *May 9, 2001*

A table of Laplace transforms and a table of Fourier series are attached. You may use your own calculator. You may also use a summary (two sides of an $8\frac{1}{2}'' \times 11''$ sheet of paper, with notes in your writing). You may not use anything else. You may not pass a calculator or summary to another person.

Show all your work. Erase or cross out any work you do not want graded. There are 6 questions on 8 pages plus the bonus question page and two pages of tables.

1. (20 points) Solve

$$y'' + 16y = 7\,\delta(t - \frac{\pi}{3}), \qquad y(0) = -2,\ y'(0) = 1.$$

2. (20 points) Let

$$A := \frac{1}{65}\begin{bmatrix} 312 & 13 & 143 & 26 & -13 \\ -177 & 432 & 127 & 4 & 23 \\ 398 & -3 & -23 & 74 & 3 \\ 573 & 167 & 197 & 19 & -167 \\ 215 & 90 & 170 & -205 & 365 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ -1 & 2 & 1 & -1 & 0 \\ 1 & 1 & 0 & 3 & -1 \\ 1 & 2 & 0 & 1 & 4 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}.$$

Then

$$B^{-1}AB = \begin{bmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Find the general solution of $\mathbf{x}' = A\mathbf{x}$.

3. (20 points) Solve:

$$\mathbf{x}''(t) + \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}^2 \mathbf{x}(t) = \mathbf{0},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{x}'(0) = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

(**Hint:** Suppose D is a diagonal matrix. How would you solve $\mathbf{y}'' + D^2\mathbf{y} = \mathbf{0}$?)

4. (15 points) Solve:

$$
\begin{aligned}
&u_t = u_{xx}, \qquad 0 < x < \pi,\ t > 0, \\
&u(0,t) = 0 = u(\pi,t), \qquad t > 0, \\
&u(x,0) = 73\sin 5x - \sqrt{591}\sin 13x, \qquad 0 < x < \pi.
\end{aligned}
$$

5. (20 points) The undamped pendulum satisfies the equations

$$x' = y$$
$$y' = -\sin x$$

where $x = \theta$ is the angle the pendulum makes with the downward vertical direction and $y = \theta'$ is the angular velocity.

(a) Find all equilibrium points of the system.

(b) Show that the energy of the pendulum

$$E(x, y) = \frac{1}{2}y^2 + 1 - \cos x$$

is constant on trajectories of the system.

(c) Show that $E(\pi, 0) = 2$ and $E(0, 2) = 2$.

(d) Consider the following snapshot of Chuck's Maple session.
Maple Plot Warning
active1dwith(plots):
Warning, the name changecoords has been redefined
active1dsys := diff(x(t),t) = y(t), diff(y(t),t)=-sin(x(t)):
active1dnumsol := b -¿ dsolve({sys,x(0)=0,y(0)=b},{x(t),y(t)},numeric):
active1dcurve := (b,range) -¿ odeplot(dsolve({sys,x(0)=0,y(0)=b},{x(t),y(t)} ,numeric),[x(t),y(t)],range,scaling=CONSTRAINED,numpoints=300):
active1dnphase := range -¿ display({seq(curve(0.5*b,range),b=1..5)},view=[-10..10,-5..5]):
active1dnphase(0..20);

pendulum01.eps

What is wrong with the plot? What probably caused Maple to make this mistake?

6. (20 points) Consider the following snapshot of Carol's Maple session.
Maple Output Maple Plot Warning 2D Math 2D Output
active1dcrit := solve({4*(y-x),-y*(2-x-y)*(-2-x-y)},{x,y});
inert2dcrit := {y = 0, x = 0}, {y = -1, x = -1}, {y = 1, x = 1};

$$crit := \{y = 0,\, x = 0\},\, \{y = -1,\, x = -1\},\, \{y = 1,\, x = 1\}$$

active1dwith(DEtools):
active1ddfieldplot([diff(x(t),t)=4*(y(t)-x(t)), diff(y(t),t)=-y(t)*(2-x(t)-y(t))*(-2-x(t)-y(t))], [x(t),y(t)],t=0..1,x=-2..2,y=-2..2,arrows=SLIM,axes=BOXED);

p601.eps

active1dwith(linalg):
Warning, the name adjoint has been redefined
Warning, the protected names norm and trace have been redefined and unprotected
active1djac := [[diff(4*(y-x),x),diff(4*(y-x),y)], [diff(-y*(2-x-y)*(-2-x-y),x),diff(-y*(2-x-y)*(-2-x-y),y)]]:
active1deigenvals(subs(crit[1],jac));
inert2d-4, 4;

$$-4,\, 4$$

active1deigenvals(subs(crit[2],jac));
inert2d-4+4*I, -4-4*I;

$$-4 + 4\,I,\, -4 - 4\,I$$

active1deigenvals(subs(crit[3],jac));
inert2d-4+4*I, -4-4*I;

$$-4 + 4\,I,\, -4 - 4\,I$$

(a) What system of differential equations is Carol studying?

(b) What are the critical points of the system?

(c) Give its type and stability of each critical point.

(d) For each critical point, give an initial condition (x_0, y_0) different from the critical point such that the corresponding solution $(x(t), y(t))$ converges to this critical point.

Bonus question (10 points)
Find:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

TABLE OF FOURIER TRANSFORMS

The functions in the table are defined for $-\pi < x < \pi$.

function	*Fourier series*
x	$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$
$\|x\|$	$\frac{\pi}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos nx$
x^2	$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$
x^3	$2\sum_{n=1}^{\infty} (-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right) \sin nx$
$f(x)$	$\sum_{n=1}^{\infty} \frac{2}{n\pi} (1-(-1)^n) \sin n\pi$

The function f in the table is equal to 1 if $0 < x < \pi$ and -1 if $-\pi < x < 0$.