Math 325

**Take-home Problem for Final Exam**, 35 points

Due at 4:15 p.m., Wednesday, May 9, 2001

You may consult your course notes, homework, and the textbooks. The problem requires using Maple. You may not consult any other books or notes. You may not discuss the exam with anyone except Prof. Stanton.

In this problem you will investigate the effect of a periodic, possibly discontinuous, forcing function on a second order linear equation with constant coefficients. Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0,\ y'(0) = 1. \tag{1}$$

The general solution of the associated homogeneous equation is

$$y(t) = A\cos t + B\sin t$$

which is periodic with period $2\pi$. Recall that the phenomenon of *resonance*, that is, solutions which are unbounded as $t \to \infty$, occurs when the forcing function $h(t)$ is a linear combination of $\cos t$ and $\sin t$. (See p. 202 in Boyce and DiPrima for a discussion of resonance.) You will investigate whether resonance occurs when the forcing function is periodic of period $2\pi$ but not necessarily continuous. The problem is similar to Problem Set E #17 in *Differential Equations with Maple*.

*A note about terminology:* If you are asked to **explain** an answer, you are being asked to give a mathematical justification, that is, a mathematical proof that the answer is correct.

(a) Using step functions, define a function $H(t)$ a whose value is $t$ on $[0, 2\pi)$, $t - 2\pi$ on $[2\pi, 4\pi)$, $t - 4\pi$ on $[4\pi, 6\pi)$ and so on. Define a Maple function $h(t)$ which agrees with $H(t)$ on the interval $[0, 10\pi]$. Plot the Maple function on the interval $[0, 30]$. It should have the appearance of a sawtooth wave. (Figure 6.3.8 on p. 316 of Boyce and DiPrima shows a sawtooth wave which has period 1.)

(b) Use **dsolve** with **method=laplace** to solve equation (1) with the function $h(t)$ defined in part (a). Explain why the solution will agree on the

interval $[0, 30]$ with the one having $H(t)$ forcing function. Plot the solution together with $h(t)$ on the interval $[0, 30]$. Do you see resonance? Compute and plot a numerical solution to confirm your answer.

(c) In part (a), you constructed a forcing function $H(t)$ with period $2\pi$. The function $H(t/2)$ has period $4\pi$. Repeat part (b) using the forcing function $h(t/2)$. Do you see resonance? If necessary, use a larger interval.

(d) Repeat part (b) using the forcing function $h(2t)$. Do you see resonance? What is the period of $H(2t)$?

(e) What can you conclude about the resonance effect for discontinuous forcing functions? Would you expect resonance to occur In equation (1) for *any* forcing function of period $2\pi$? (*Hint*: Is the function $H(2t)$ periodic with period $2\pi$?) When do you expect resonance to occur for piecewise continuous $2\pi$ periodic forcing functions? Explain your answer. (*Hint*: Consider the $2\pi$ periodic Fourier series of the forcing function. You might find it useful to look at the section of problems on *Periodic Forcing Terms* on p. 563 of Boyce and DiPrima.) You might try some other periodic forcing functions to check your answer.

**Bonus question** (10 points): This winter and spring, the remarkable exhibit *Witness & Legacy: Contemporary Art About The Holocaust* has been at the Warner Gallery of the South Bend Regional Museum of Art (at Century Center, in downtown South Bend). In conjunction with the exhibit, more than twenty cultural institutions and colleges and universities in the area, including the University of Notre Dame, have sponsored events as part of *In Unison: The Collaborative Community Programs of Witness and Legacy*. Here is the description of the exhibit from the museum's web page (http://www.sbt.infi.net/~sbrma):

> The South Bend Regional Museum of Art (SBRMA) and the Kurt and Tessye Simon Fund for Holocaust Remembrance are sponsoring an important and provocative exhibition entitled Witness & Legacy: Contemporary Art About The Holocaust. Organized by the Minnesota Museum of American Art and the Regis Foundation, this moving exhibition presents the work of twenty-two American artists who include Holocaust survivors, "second generation" children of survivors and others drawn to the subject matter by their own humanitarianism and empathy. The artworks themselves incorporate a diversity of forms in

a variety of media including painting, sculpture, photography, graphic design, and multimedia installation.

The purpose of this bonus question is to encourage you to see the exhibit. I recommend that you allow at least an hour. I can't claim that the exhibit has anything to do with differential equations, but there's at least a little related to mathematics in it. To get credit for the bonus question, you have to see the exhibit, which is open Tuesday, Wednesday and Friday 11a.m.-5p.m., Thursday 11a.m.-9p.m., and Saturday and Sunday noon-5p.m., and write a brief paragraph about it which mentions at least one mathematical thing you found in the exhibit.