Name: ________________________ **Professor:** ____________
tion: ____________

Mathematics 325
Fall Semester 2002
Final Exam.
December 20, 2002

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1. This Examination contains 25 problems worth a total of 150 points. The test will be exactly 120 minutes in length. 2. For each multiple choice question, please mark an X on the correct answer on the answer sheet. **Do not circle it.** 3. Calculators, books and notes are not allowed. 4. A table of Laplace transforms and a table giving type and stability properties in terms of eigenvalues are included at the end of the booklet. 5. Hand in the entire test.

Sign the pledge: "On my honor, I have neither given nor received unauthorized aid on this Exam":
________________________ **HAVE A NICE VACATION**

1. The Wronskian of $\{1, \cos t, \sin t\}$ is (a). $\cos t$ (b). $\sin t$ (c). 1 (d). -1 (e). $\sin^2 t - \cos^2 t$

2. Determine a suitable form for a particular solution of the equation

$$y'''' - 3y'' - 4y = \cos 2t.$$

(a). $A \sin 2t$ (b). $A \cos 2t + B \sin 2t$ (c). $B \cos 2t$ (d). $(At + B) \cos 2t$ (e). $At \cos 2t + Bt \sin 2t$

3. Find the general solution of the equation

$$y''' + y'' - y' - y = 0.$$

(a). $C_1 e^t + C_2 e^{-t} + C_3 te^{-t}$ (b). $C_1 e^t + C_2 e^{-t} + C_3 e^{-t}$ (c). $C_1 e^t + C_2 te^t + C_3 e^{-t}$ (d). $C_1 e^{-t} + C_2 te^{-t} + C_3 t^2 e^{-t}$ (e). $C_1 e^t + C_2 e^{-t}$

4. Let $X_1(t) = 3 + t,\ X_2(t) = -t + 2t^2,\ X_3(t) = 2 - t^2, X_4(t) = 3 - t + 4t^2$. Which of the following is *false*? (a). X_1, X_2, X_4 are linearly independent (b). X_1, X_2, X_3 are linearly independent (c). X_1, X_2 are linearly independent (d). X_1, X_3 are linearly independent (e). X_2, X_3, X_4 are linearly independent

5. Let

$$f(t) = \int_0^t (t - \tau)^3 e^{2\tau} d\tau.$$

Find the Laplace transform of f. (a). $\frac{6}{s^4(s+2)}$ (b). $\frac{6}{s^4(s-2)}$ (c). $s^3 e^{-2s}$ (d). $\frac{3}{s^4(s+2)}$ (a). $\frac{6}{(s+2)^4}$

6. The Laplace transform of

$$f(t) = t^2 \delta(t - 3)$$

is (a). $\frac{2}{s^3}$ (b). $9e^{-3s}$ (c). e^{-3s} (d) $\frac{2}{(s-3)^3}$ (e). $3e^{-9s}$.

7. Solve the initial value problem

$$y'' = u_1(t),\ \ y(0) = 1, y'(0) = 0.$$

(a). $y = 1 + 2u_1(t)t^2$ (b). $y = 1 + u_1(t)(t-1)^2$ (c). $y = 1 + \frac{1}{2}u_1(t)t^2$ (d). $y = 1 + \frac{1}{2}u_1(t)(t-1)^2$ (e).

$y = \frac{1}{2}(1 + u_1(t)t^2)$

8. Solve the initial value problem

$$y'' - 4y = 8\delta(t-1), \quad y(0) = y'(0) = 0.$$

(a). $y = 2(e^{2t} + e^{-2t})$ (b). $y = 2u_1(t)(e^{2t} + e^{-2t})$ (c). $y = 2u_1(t)(e^{2t} - e^{-2t})$ (d). $y = 2u_1(t)(e^{2(t-1)} + e^{-2(t-1)})$ (e). $y = 2u_1(t)(e^{2(t-1)} - e^{-2(t-1)})$

9. Consider the initial value problem

$$y' = 5y,\ \ y(0) = 1.$$

Using step size $h = \frac{1}{10}$ and two steps of the (forward) Euler method, what is the error at $t = \frac{1}{5}$ compared to the actual solution to the equation? (a). $|1 - \frac{9}{4}|$ (b). $|e - \frac{9}{4}|$ (c). $|1 - \frac{3}{2}|$ (d). $|e - \frac{3}{2}|$

(e). $|1 - e|$

10. Find the general solution of

$$\frac{d\mathbb{X}}{dt} = (\,1\,53 - 1\,)\,\mathbb{X}.$$

(a).

$$C_1 e^{4t}\,(\,53)+C_2 e^{-4t}\,(\,1-1\,)\,.$$

(b).

$$C_1 e^{4t}\,(\,53)+C_2 e^{4t}\,(\,1-1\,)\,.$$

(c).

$$C_1 e^{4t}\,(\,53)+C_2 e^{-4t}\,(\,0-1\,)\,.$$

(d).

$$C_1 e^{4t}\,(\,35)+C_2 e^{-4t}\,(\,11)\,.$$

(e).

$$e^{4t}\,(\,1-1\,)\,.$$

11. The trajectory corresponding to the solution of

$$\frac{dx}{dt} = 4y, \quad x(0)=1, \frac{dy}{dt} = -x, \quad y(0)=0$$

is (a). $x^2+4y^2=1$, traveling clockwise. (b). $4x^2+y^2=1$, traveling clockwise. (c). $x^2+4y^2=1$, traveling counterclockwise. (d). $4x^2+y^2=1$, traveling counterclockwise. (e). $x^2+4y^2=4$, traveling counterclockwise.

12. The matrix $(1\,221)$ has eigenvalues 3 and -1 and eigenvectors (11) and $(1-1)$ respectively. Find the general solution to $\frac{d\mathbb{X}}{dt} = (1\,221)\,\mathbb{X} + \left(0e^t\right)$.

(a).

$$C_1e^t\,(10) + C_2e^{-t}\,(01) + e^{3t}\,(11)\,.$$

(b).

$$C_1e^{3t}\,(11) + C_2e^{-t}\,(1-1) - \frac{1}{2}e^t\,(01)\,.$$

(c).

$$C_1e^{3t}\,(11) + C_2e^{-t}\,(1-1) + \frac{1}{2}e^t\,(01)\,.$$

(d).

$$C_1e^{3t}\,(11) + C_2e^{-t}\,(1-1) + \frac{1}{2}e^t\,(10)\,.$$

(e).

$$C_1e^{3t}\,(11) + C_2e^{-t}\,(1-1) - \frac{1}{2}e^t\,(10)\,.$$

13. The system of nonlinear differential equations:

$$\frac{dx}{dt} = x - xy\frac{dy}{dt} = -y + xy$$

has an isolated critical point at $(1,1)$. The linearization of this system around the critical point $(1,1)$, using the coordinates $u = x-1$, $v = y-1$, is

(a).

$$\frac{d}{dt}\,(uv) = (1\,00-1)\,(uv)$$

(b).

$$\frac{d}{dt}\,(uv) = (0\,1-10)\,(uv)$$

(c).

$$\frac{d}{dt}\,(uv) = (1\ -111)\,(uv)$$

(d).

$$\frac{d}{dt}\,(uv) = (0\ -110)\,(uv)$$

(e).

$$\frac{d}{dt}\,(uv) = (0\,110)\,(uv)$$

14. Consider the nonlinear system of differential equations

$$\frac{dx}{dt} = y + x^2 \frac{dy}{dt} = -\sin x - y.$$

Classify the type and stability of the critical point at $(0,0)$.

(a). $(0,0)$ is a spiral point and is unstable. (b). $(0,0)$ is an improper node and unstable. (c). $(0,0)$ is a saddle point and is unstable. (d). $(0,0)$ is a center and is stable. (e). $(0,0)$ is a spiral point and is asymptotically stable.

15. Find the eigenvalues of the matrix $(\,0\,2\,16 - 2\,0\,9\,0\,0\,11)$

(a). $2i$, -$2i$, 11 (b). 2, 2, 11 (c). $2i$, -$2i$, 16 (d). 2, -2, 11 (e). $2i$, -$2i$, $3i$

16. Find all critical points for the following system of nonlinear differential equations:

$$\frac{dx}{dt} = x - x^2 - xy \frac{dy}{dt} = 2y - y^2 - 3xy$$

(a). $(0,0)$ (b). (0,0), (1,0) (c). $(0,0),(1,0),(0,2)$ (d). $(0,0),(1,0),(0,1),(0,2)$ (e). $(0,0),(1,0),(0,2),(\frac{1}{2},\frac{1}{2})$

17. Use the Liapunov function $V(x,y) = x^2 + y^2$ to decide the stability of the critical point (0,0) of the system

$$\frac{dx}{dt} = x^3 - y^3 \frac{dy}{dt} = xy^2 + 4y^3.$$

(a). $(0,0)$ is unstable. (b). $(0,0)$ is asymptotically stable. (c). $(0,0)$ is stable but not asymptotically stable. (d). $(0,0)$ is a center and is stable. (e). One cannot conclude the stability from the Liapunov function.

18. Which of the following functions is not periodic? (a). $\sin \pi x$ (b). $(\cos x)^2$ (c). $e^x - e^{-x}$ (d). $e^{\cos x}$ (e). 100

19. Let $f(x) = x^3 + 1$ on $-1 < x \leq 1$ and $f(x+2) = f(x)$. What value does the Fourier series of f converges to at $x = 1$?

(a). 2 (b). 0 (c). 1

(d). -1 (e). The Fourier series diverges at $x = 1$. 20. What is the solution of the heat equation

$$\{4u_{xx} = u_t, \qquad 0 < x < 2,\ t > 0 u(0,t) = u(2,t) = 0, \quad 0 < t u(x,0) = \sin \pi x, \quad 0 < x < 2$$

(a).

$$u = e^{\frac{-\pi^2 t}{2}} \sin \pi x.$$

(b).

$$u = e^{-4\pi^2 t} \sin \pi x.$$

(c).

$$u = e^{\frac{-\pi^2 t}{4}} \sin \pi x.$$

(d).

$$u = e^{\frac{-\pi^2 t}{4}} \sin \frac{\pi x}{2}.$$

(e).

$$u = e^{4\pi^2 t} \sin \pi x.$$

21. Consider the two-point boundary value problem

$$y'' + 9y = 0,\ y(0) = 1,\ y(\pi) = \alpha.$$

For which value(s) of α does this problem have infinitely many solutions? (a). 0 (b). -1 (c). 1

(d). 1 or -1 (e). No α will satisfy the conditions. 22. Let $f(x) = x,\ 0 \le x < 1$. Consider the even

extension of f and $f(x+2) = f(x)$. The Fourier coefficient a_1 in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

is (a). 0 (b). $-\frac{2}{\pi}$ (c). $\frac{2}{\pi}$ (d). $-\frac{4}{\pi^2}$ (e). $\frac{4}{\pi^2}$ 23. Consider the equation

$$u_{xx} - u_{tt} = 0.$$

Let $u = X(x)T(t)$ be a solution. Then X and T satisfy which of the following pairs of ordinary differential equations: (a). $X' + \lambda X = 0,\ T' + \lambda T = 0$ (b). $X'' + \lambda X = 0,\ T'' - \lambda T = 0$ (c).

$X'' - \lambda X = 0,\ T' + \lambda T = 0$ (d). $X'' + \lambda X = 0,\ T'' + \lambda T = 0$ (e). $X'' + \lambda X = 0,\ T'' + \lambda^2 T = 0$

24. Let

$$f(x) = \{3, \qquad 0 < x < 10, \qquad 1 < x < 2.$$

Consider the odd extension of $f(x)$ with $f(x+4) = f(x)$. The Fourier coefficient a_5 in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

is

(a). $\frac{1}{5\pi}$ (b). $\frac{2}{5\pi}$ (c). $\frac{4}{5}$ (d). 0 (e). $\frac{-2}{5\pi}$

25. The sine series of $f(x) = x$, $0 \le x \le 2$ is

$$\sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{2}.$$

What is the solution of the wave equation

$$\{u_{xx} = u_{tt}, \qquad 0 < x < 2,\ t > 0u(0,t) = u(2,t) = 0, \quad t > 0u(x,0) = x,\ u_t(x,0) = 0, \quad 0 < x < 2$$

(a).

$$u = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{2} \sin\frac{n\pi t}{2}.$$

(b).

$$u = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{2} \cos\frac{n\pi t}{2}.$$

(c).

$$u = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{2} \sin n\pi t.$$

(d).

$$u = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\pi^2 n^2 t} \sin\frac{n\pi x}{2}.$$

(e).

$$u = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{\frac{-\pi^2 n^2 t}{4}} \sin 2n\pi x.$$