Math 335–Final Exam
December 1992

1. The Archimedean Property for the real numbers **R** states that
   (a) If $a > 0$ and $b > 0$, then there exists some positive integer n such that $na > b$.
   (b) If $a > 0$ and $b > 0$ then $ab > 0$.
   (c) If $a > 0$ then there exists a positive integer n such that $\frac{1}{a} > n$.
   (d) the completeness axiom fails for the set **Z** of integers.
   (e) If $a > 0$ and $b > 0$ then there exists a positive integer n such that $a^n > b$.

2. The completeness axiom for the set **R** of real numbers states that
   (a) any nonempty subset of **R** that is bounded above has a supremum.
   (b) If S is a bounded subset of **R** then the supremum of S is the maximal element of S.
   (c) If $S \subset \mathbf{R}$ is bounded below then S has a supremum.
   (d) If $S \subset \mathbf{R}$ is unbounded then S has no supremum.
   (e) If a set $S \subset \mathbf{R}$ has no supremum, then S is empty.

3. Only one of the following relations below is true for all integers $n \geq 1$. Which one is it?
   (a) $7^n - 6n - 1$ is divisible by 36
   (b) $1 + 2 + \cdots + n = \frac{n(n-1)}{2}$
   (c) $\sum_{k=1}^{n} (0.99999)^k \leq 100$
   (d) $\sum_{k=1}^{n} k! = (n+1)!$
   (e) $1^2 + 2^2 + \cdots + n^2 = (n+1)(n+2)(n+3)$

4. Let f be a real-valued function whose domain is the set of all real numbers. Suppose that f is **NOT** continuous at a. Which one of the following statements is then necessarily true?
   (a) There exists $\epsilon > 0$ such that for any $\delta > 0$ there exists an x with $\mid x - a \mid < \delta$ and $\mid f(x) - f(a) \mid \geq \epsilon$.
   (b) For any sequence (x_n) with $\lim x_n = a$, $\lim f(x_n) \neq f(a)$.
   (c) For any $\epsilon > 0$ there exists $\delta > 0$ such that $\mid x - a \mid < \delta$ implies $\mid f(x) - f(a) \mid \geq \epsilon$.
   (d) For any $\epsilon > 0$ and $\delta > 0$ there exists an x such that $\mid x - a \mid < \delta$ and $\mid f(x) - f(a) \mid \geq \epsilon$.
   (e) There exists a sequence (x_n) such that $\lim x_n = a$ and $\lim f(x_n) = \infty$.

5. Consider the following three statements about sequences:
   $\langle 1 \rangle$ Every monotone sequence has a bounded subsequence.
   $\langle 2 \rangle$ Every convergent sequence is bounded.
   $\langle 3 \rangle$ Every bounded sequence has a Cauchy subsequence.
   Which of these statements are true?
   (a) $\langle 2 \rangle$ and $\langle 3 \rangle$ (b) $\langle 1 \rangle$ and $\langle 3 \rangle$ (c) $\langle 1 \rangle$ and $\langle 2 \rangle$ (d) only $\langle 1 \rangle$ (e) only $\langle 3 \rangle$

6. Consider the sequence (f_n) of functions on the interval $[0, 2]$ given by $f_n(x) = \frac{x^n}{2^n}$. Which **one** of the following statements accurately describes the behavior of this sequence?
   (a) The sequence converges pointwise to a function which is not continuous.
   (b) The sequence converges pointwise to a continuous function, but does not converge uniformly.
   (c) The sequence does not converge pointwise to any function.
   (d) The sequence converges uniformly to a function which is not continuous.
   (e) The sequence converges uniformly to a continuous function.

7. Let (s_n) be the sequence given by $s_n = \frac{2^{1/n} + 4^{-n}}{n^{1/n} + 5n}$. Then $\lim s_n$ is
   (a) 0 (b) ∞ (c) 1 (d) 1/5 (e) 2/5

8. Which one of the following functions on $(0, 1]$ can be extended to a continuous function on $[0, 1]$?

(a) $f(x) = x\cos(1/x)$ (b) $f(x) = \dfrac{\cos x}{x}$ (c) $f(x) = \cos(1/x)$ (d) $f(x) = \dfrac{x+1}{x^3 + 2x^2 + x}$

(e) $f(x) = \max(x, 1/x)$

9. Which one of the following statements is true?

(a) Any uniformly continuous function on $(2, 7)$ is bounded.

(b) Any bounded continuous function on $(2, 7)$ is uniformly continuous.

(c) Any one-to-one (i.e. injective) continuous function on $(2, 7)$ is bounded.

(d) If $f(x)$ is continuous on $(2, 7)$ then there exists $x \in (2, 7)$ with $f(x) = 5$.

(e) if $f(x)$ is continuous on $(2, 7)$ then there exists a sequence (x_n) of points from $(2, 7)$ such that $\lim x_n = 2$ and $\{f(x_n) : n \geq 0\}$ is bounded.

10. Suppose that $\sum a_n$ and $\sum b_n$ are convergent series with $a_n > 0$ and $b_n > 0$. Which **one** of the following four series is necessarily convergent? (Hint: You might try using the comparison test, or search for examples among the class of p-series.)

(a) $\sum a_n b_n$ (b) $\sum \dfrac{a_n}{b_n}$ (c) $\sum \sqrt{a_n}$ (d) $\sum \dfrac{1 - a_n}{1 + b_n}$ (e) $\sum \dfrac{1}{\sqrt{a_n}}$

11. Suppose that f and g are real-valued functions with domain the set of all real numbers. Which **one** of the following statements is **FALSE**? (Note that $f \circ g$ denotes the composite of f with g.)

(a) If f is continuous and $f \circ g$ is continuous then g is continuous.

(b) If f is continuous and g is continuous then $f + g + f \circ g$ is continuous.

(c) If f and g are continuous then $\min(f, g)$ is continuous.

(d) If f and g are continuous then $| fg |$ is continuous.

(e) If $f(x)$ is continuous then $\sqrt{| f |}$ is continuous.

12. Consider the series $\sum \dfrac{x^n}{2^n(n+1)^2}$. What is its radius of convergence?

(a) 2 (b) 1 (c) 0 (d) ∞ (e) 4

13. Recall that the *interval of convergence* of a power series is the set of numbers x for which the power series converges. What is the interval of convergence for the series $\sum \dfrac{x^n}{n}$?

(a) $[-1, 1)$ (b) $(-\infty, \infty)$ (c) $(-1, 1)$ (d) $(0, \infty)$ (e) $[-1, 1]$

14. Let $(s_n)_{n \geq 1}$ be the sequence defined inductively by setting $s_1 = 1$ and $s_{n+1} = \dfrac{s_n}{2} + \dfrac{2}{s_n}$. You may take it for granted that this sequence converges. The limit of the sequence is

(a) 2 (b) $\sqrt{2}$ (c) 4 (d) 0 (e) 1

15. Which **one** of the following series converges? (Note that, for a real number x, the notation $\lfloor x \rfloor$ denotes the greatest integer with is less than or equal to x.)

(a) $\sum (-1)^n \dfrac{1}{\lfloor n\pi \rfloor}$ (b) $\sum \dfrac{1}{\sqrt{n}\log n}$ (c) $\sum (-1)^n \sin(\dfrac{n\pi}{6})$ (d) $\sum (-1)^n \dfrac{n!}{2^n}$

(e) $\sum \lfloor 2^n/n^2 \rfloor$