Math 335 Real Analysis

Exam II

October 17, 2001

Answer all 5 questions. All questions have equal points. Symbols and markings without complete sentences will not be considered as answers.

1. State the Intermediate Value Theorem (I.V.T.).
   a) State clearly a consequence of the IVT for any odd degree polynomial $P(x)$.
   b) Specify a closed interval $[a, b]$ on which the polynomial
   $P(x) = 8x^3 - 36x^2 + 46x - 15$
   must have a root. Give your reasons.

2. Let S be a subset of $\mathbb{R}$.

   a) Define what is meant by saying S is bounded above.

   b) Define what is meant by saying U is a least upper bound of S.

   c) State a theorem which guarantees the existence of a least upper bound.

   d) Give the least upper bound and the greatest lower bound of the set formed by the terms of the sequence $\{S_n\} = \{(-1)^n + 1 + \frac{1}{n}\}_{n=1}^{\infty}$

3. The double inequality $-1 \leq \sin \frac{1}{x} \leq 1$ holds for all $x \neq 0$. Use the Sandwich Theorem to show $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

   Determine whether the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   has a derivative at $x = 0$ and if it does have a derivative, determine its value.

4. Let f be a function defined on an open interval I which has an absolute maximum value occurring at $x_0 \epsilon I$. If $f'(x_0)$ exists, prove that $f'(x_0) = 0$.

5. State the Mean Value Theorem (M.V.T.). By carefully applying the M.V.T., show that $\sin x > x \cos x$ when $0 < x < \pi$.
   Hint: Take $f(x) = \sin x - x \cos x$.