Math 335 Real Analysis

Exam III

November 19, 2001

Answer all 4 questions. All questions are marked with their point value. Symbols and markings without complete sentences will not be considered as answers.

1. (40 pts.) Beginning with a bounded function f defined on a closed interval $[a, b]$, write down all of the steps leading to the definition of the upper and lower Darboux integrals of f and the notion of Darboux integrability.

   [Your description should be clear on the meaning of the terms subdivision $\triangle$, the I_i, m_i and M_i, the upper and lower sums of f with respect to each subdivision $\triangle$, the comparison of any two such sums, the definition of the upper and lower Darboux integrals and, finally, the definition of integrability.]

2. (20 pts.) If $1 + x > 0$, show that $\log\,(1 + x) \leq x$. (Hint: consider $f(x) = \log(1 + x) - x$).

3. (20 pts.)

a) State any theorem you know guaranteeing the Darboux integrability of functions on closed intervals.

b) Let $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational.} \end{cases}$

Show that $\underline{\int_0^1} f(x)dx = 0$ and $\overline{\int_0^1} f(x)dx = 1$

c) Give an example of a function on [0, 1] that is <u>not</u> *Riemann integrable.*

4. (20 pts.)

a) Let f be a function on $[a, b]$ and $\triangle$ a subdivision of $[a, b]$. What is meant by a Riemann sum of f with respect to $\triangle$?

b) What does it mean to say that a real number A is the Riemann integral of f on $[a, b]$?

c) Is the function defined by

$$f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}$$

Riemann integrable on [0, 1]. Why?