Math 335 Real Analysis

Exam I

December, 2001

Answer all 5 questions. All questions have equal points. Symbols and markings without complete sentences will not be considered as answers.

1. Let f be a function defined on $\mathbf{R}$ and L a real number.

   a) Define what is meant by $f(x) \rightarrow L$ as $x \rightarrow a$.

   b) Define what is meant by saying f is continuous at $x = a$.

   Let $f(x) = x^2 + 1$.
   Find δ so that $\mid f(x) - f(o) \mid < \epsilon$ when $\mid x - o \mid < \delta$. Your answer will give δ in terms of ϵ.

4 The double inequality $-1 \leq \sin \frac{1}{x} \leq 1$ holds for all $x \neq 0$. Use the Sandwich Theorem to show $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$.

Determine whether the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has a derivative at $x = 0$ and if it does have a derivative, determine its value.

State the Intermediate Value theorem

5. Over the course of one hour the temperature T of an iron bar lies in the range $0°C$ to $1°C$. Show that at some time (measured in hours) the temperature $T(t)$ and five t must coincide, i.e. $T(\epsilon) = t$ for some $t \epsilon [0, 1]$.

   Hint: Consider the continuous function $f(t) = T(\epsilon 1 - t$ or $[0, 1]$.

6. (20 pts.)

   a) Let f be a function on $[a, b]$ and $\triangle$ a subdivision of $[a, b]$. What is meant by a Riemann sum of f with respect to $\triangle$?

   b) What does it mean to say that a real number A is the Riemann integral of f on $[a, b]$?

   c) Is the function defined by
   $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}$
   Riemann integrable on [0, 1]. Why?

7. Let S be a set and $d(x, y)$ a number associated to each pair (x, y) with $x, y \epsilon S$. Write down the conditions $i), ii), III)$ that d must satisfy in order to be a metric on S.

8. Let S be a metric space with metric d.

a) Define what is meant by saying that a subset A is open.

b) Define what is meant by saying that a subset B is closed.

c) Show that if A is open, then $B = S - A$ is closed.

9. Give an example of a set A in the metric space $\mathbb{R}^2$ (with the euclidean metric) which is neither open nor closed.

   a) Indicate why it is not open.

   b) Indicate why it is not closed.