1. (20 pts) The following strange inequality turns out to be true:

$$\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\,dx \le \frac{16\pi^3}{3}\,.$$

Find a proof of this inequality. You might have some success using general properties of the integral, especially properties that involve inequalities and/or absolute values.

2. (30 pts) Let (F_n) be a decreasing sequence (i.e., $F_1 \supset F_2 \supset F_3 \cdots$) of closed bounded nonempty sets in $\mathbf{R}^n$. Prove that the intersection $F = \cap_{n=1}^{\infty} F_n$ is also closed, bounded and nonempty.

3. Let (f_n) be the sequence of functions given by $f_n(x) = e^{-n(x^2+1)}$. Note that (f_n) converges uniformly on $[0,1]$. Calculate $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

(a) 0 (b) $e/2$ (c) e (d) ∞ (e) 1

4. Calculate the derivative of the function $f(x) = (\sin x)^{\cos x}$, x in $(-\pi/2, \pi/2)$. Be careful!

(a) $f'(x) = \ln(\sin x)\,(\sin x)^{\cos x} + (\sin x)^{1+\cos x}$ (b) $f'(x) = \cos^2 x\,(\sin x)^{(-1)+\cos x}$

(c) $f'(x) = \cos^2 x\,(\sin x)^{(-1)+\cos x} - \ln(\sin x)\,(\sin x)^{1+\cos x}$

(d) $f'(x) = \cos^2 x\,(\sin x)^{1+\cos x} - \ln(\sin x)\,(\sin x)^{(-1)+\cos x}$

(e) $f'(x) = -\cos x\,(\sin x)^{\cos x}$

5. Let $f(x)$ be the function defined by the following rule:

$$f(x) = \begin{cases} 2x, & \text{if } x \le 1; \\ -x, & \text{if } x > 1. \end{cases}$$

Consider the function $g(x) = \int_0^x f(t)\,dt$. Which **one** of the following statements about g is true?

(a) $g(x)$ is defined and differentiable on $(0,\infty)$, but $g'(x)$ is not continuous on $(0,\infty)$.

(b) $g(x)$ is defined and continuous on $(0,\infty)$, but is not differentiable for $x = 1$.

(c) $g(x)$ is defined and differentiable on $(0,\infty)$, and $g'(x)$ is continuous on $(0,\infty)$.

(d) $g(x)$ is defined for all $x > 0$, but is not continuous at $x = 1$.

(e) $g(x)$ is not defined for $x > 1$ because the function f is not integrable on $[0,x]$ for $x > 1$.

6. Let E be the set of rational numbers, considered as a subset of the metric space $\mathbf{R}$ of real numbers. Which **one** of the following statements about E is true.

   (a) The interior of E is the empty set, the closure of E is E itself, and the boundary of E is E itself.

   (b) The interior of E is $\mathbf{R}$, the closure of E is $\mathbf{R}$, and the boundary of E is the empty set.

   (c) The interior of E is the empty set, the closure of E is $\mathbf{R}$, and the boundary of E is $\mathbf{R}$.

   (d) The interior of E is E itself, the closure of E is $\mathbf{R}$, and the boundary of E is the set of irrational numbers.

   (e) The interior of E is the empty set, the closure of E is $\mathbf{R}$, and the boundary of E is the set of irrational numbers.

7. Which **one** of the following statements about the Darboux integral is **FALSE**?

   (a) Every continuous function on $[a, b]$ is integrable.

   (b) Every integrable function on $[a, b]$ is continuous.

   (c) A bounded function f on $[a, b]$ is integrable if an only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

   (d) Any integrable function f on $[a, b]$ is bounded.

   (e) A bounded function f on $[a, b]$ is integrable if and only if the supremum of the lower Darboux sums for f on $[a, b]$ is equal to the infimum of the upper Darboux sums for f on $[a, b]$.

8. Let $f(x) = (x-1)^3 + \dfrac{1}{3}(x+1)^3$. Compute the degree two Taylor expansion of $f(x)$ around the point $x = 0$.

   (a) $-\dfrac{2}{3} + 4x - 2x^2$ (b) $\dfrac{2}{3} - 3x + 3x^2$ (c) $\dfrac{2}{3} - \dfrac{x}{3} + \dfrac{x^2}{6}$ (d) $\dfrac{2}{3} + 6x$

   (e) $\dfrac{2}{3} + \dfrac{2x}{3}$

9. Consider the function $f(x) = x^3$, which is continuous and increasing on $(-\infty, \infty)$. Let $g(y)$ be the inverse function to f. Which **one** of the following statements about these functions is true?

   (a) Although f takes on all real values, its inverse g is only defined for $y \geq 0$.

   (b) $g'(1) = 2/3$

   (c) Since f is an increasing function, its inverse g is a decreasing function.

   (d) Since the function f is differentiable, so is its inverse g.

   (e) The function $g(y)$ is continuous but is not differentiable for $y = 0$.

10. Which **one** of the following inequalities is true? (You might try for instance proving these inequalities with the Mean Value Theorem.)

    (a) $|\cos x - \cos y| \leq |x-y|^2$ for x, y in $\mathbf{R}$.

    (b) $|\sin x - \sin y| \geq |x-y|$ for x, y in $\mathbf{R}$.

    (c) $e^x \geq |x|$ for x in $(-1, 0)$

    (d) $|\cos x - \cos y| \leq |x-y|$ for x, y in $\mathbf{R}$.

    (e) $\sin x \geq x$ for x in $(0, 1)$.

11. Which of the following statements gives an accurate representation of the Weierstrass Approximation Theorem?

    (a) Every bounded continuous function on $\mathbf{R}$ can be uniformly approximated by polynomials on $\mathbf{R}$.

    (b) Every continuous function on a closed interval $[a, b]$ can be uniformly approximated by polynomials on $[a, b]$.

    (c) Any uniformly continuous function on $\mathbf{R}$ can be uniformly approximated by polynomials on $\mathbf{R}$.

    (d) Any sequence of polynomial functions on a closed interval $[a, b]$ converges to a continuous function on $[a, b]$.

    (e) If a sequence of polynomials converges uniformly on a closed interval $[a, b]$, then the sequence converges to a polynomial function.

12. Consider the function $f(x) = \displaystyle\int_0^x e^{t^2}\,dt$. What is the Taylor series expansion for f around the point $x = 0$?. (Hint: first find the corresponding Taylor series expansion for $g(x) = e^{x^2}$. Note that if $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ is the Taylor expansion of a function $g(x)$, then $\displaystyle\sum_{k=0}^{\infty} a_k x^{2k}$ is the Taylor expansion for $g(x^2)$).

    (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)(n!)}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{(2n+1)!}$

    (e) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(n+1)!}$

Version 2 color:

Math 336–Final Exam
May 1993

This exam will be 2 hours in length. It consists of 2 essay questions whose point values are indicated in the body of the test, and 10 multiple-choice problems worth 10 points each. Write the answers to the essay questions in the blue book, and for each of the multiple-choice problems mark an **X** over the letter below that corresponds to the correct choice. You may use the back of the blue book for scratch work.

At the end of the exam please hand in a **complete** test booklet (answer sheet + question sheets) with your name on it, together with a blue book, also with your name on it.

Name: ____________________________

Prof: ____________________________

3.	(•)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(•)	(d)	(e)
5.	(a)	(•)	(c)	(d)	(e)
6.	(a)	(b)	(•)	(d)	(e)
7.	(a)	(•)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(•)
10.	(a)	(b)	(c)	(•)	(e)
11.	(a)	(•)	(c)	(d)	(e)
12.	(a)	(b)	(•)	(d)	(e)

Mult. Choice	______
1	______
2	______
Total	______

2a's 3b's 3c's 1d's 1e's

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