Math 336, Final Exam, Spring (1994)

The test will be 2 hours in length. Please write your name on the cover of your blue book and write the solutions inside; start the solution to each problem on a new page. This test is being administered under the provisions of the Honor Code. Your work should be your own, and you should not make use of any outside material (textbooks, notes) during the test. What you write should be neat, grammatical, clear and concise. Good luck.

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1 (30 pts) Write an essay at least two pages in length on the theory of integration. State the major definitions in the theory, describe the major theorems, and give the proof of at least one of these theorems.

2 (a) (10 pts) Define what it means for a subset of $\mathbb{R}^n$ to be compact. (b) (10 pts) Give an example of a subset of $\mathbb{R}^2$ which is **not** compact. Justify your answer.

3 Let (f_n) be a sequence of functions on the interval $[a, b]$. a (5 pts) Define what it means for the sequence (f_n) to converge **pointwise** on $[a, b]$. b (5 pts) Define what it means for the sequence (f_n) to converge **uniformly** on $[a, b]$. c (10 pts) Give an example of a sequence (f_n) which converges pointwise on $[a, b]$ but not uniformly. Justify your answer.

4 (20 pts) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Determine the largest interval on which f represents a continuous function.

5 This problem has four basic parts, worth 10 points each. There is also an extra-credit portion, worth 10 points.

(a) State Taylor's theorem.

(b) Let $n \geq 1$ be an integer. Use Taylor's theorem to show that there is a number c_n between n and $n+1$ such that

$$\ln(n+1) = \ln(n) + \frac{1}{n} - \frac{1}{2c_n^2} \ .$$

(Hint: apply Taylor's theorem to the function $f(x) = \ln(n+x)$. Observe that $f(0) = \ln(n)$, and use Taylor's theorem with the degree of the

(c) Rewrite the equation from part (b) in the form

$$\frac{1}{n} + \ln(n) - \ln(n+1) = \frac{1}{2c_n^2} .$$

By adding these equations up as n varies, show that for any $k \geq 1$ there is an equality

$$\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k+1) = \sum_{n=1}^{k} \frac{1}{2c_n^2} .$$

(d) Recall from part (b) that $c_n > n$. Using this, conclude from part (c) that the limit

$$\lim_{k\to\infty} \left[\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k+1)\right]$$

exists. (You might want to appeal to the comparison test for the convergence of series). This value of this limit is called *Euler's constant*, and is approximately 0.577216. No one knows whether or not Euler's constant is a rational number.

(e) **Extra Credit** (10 pts) Give a geometric interpretation of Euler's constant in terms of areas of region(s) in the plane.

6 (20 pts) Let $f(x)$ be the function defined by the formula

$$f(x) = \int_0^{\sin x} e^{t^2}\, dt .$$

Compute $f'(x)$. (Hint: let $g(y) = \int_0^y e^{t^2}\, dt$, and note that $f(x) = g(\sin x)$.)