Math 336, Final Exam, Spring 1995

The exam will be two hours in length. Please write your name on the cover of your blue book and write the solutions inside; start the solution to each problem on a new page. Each problem has the point value indicated. This test is being administered under the provisions of the Honor Code. Your work should be your own, and you should not make use of any outside material (textbooks, notes) during the test. What you write should be neat, grammatical, clear and concise. Good luck.

Please write your name at the top of this question sheet and return the sheet inside your exam booklet.

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1 (a) (10 pts) State Abel's Theorem. (b) (15 pts) Consider the power series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} .$$

Take for granted that this power series converges to $\log(1+x)$ for x in the **open interval** $(-1,1)$. Use this fact, together with Abel's theorem, to establish the identity

$$\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} .$$

Be sure to give a careful argument. Your argument is almost certainly incomplete if it does not use the fact (which you may also take for granted) that the function $\log(1+x)$ is continuous at $x=1$.

2 (a) (10 pts) Let (f_n) be a sequence of functions on a set S. Define what it means for the sequence (f_n) to **converge uniformly** on S to a function f. (b) (15 pts) Suppose that (f_n) is a sequence of functions on S which converges uniformly on S to a function f, and that g is a bounded function on S. (Here "bounded" means that there exists some $M > 0$ such that $|g(x)| < M$ for all $x \in S$). Using the definition from part (a), show that the sequence (gf_n) converges uniformly on S to the product function gf. (c) (10 pts **extra credit**) Show by example that (gf_n) does not necessarily converge uniformly to gf if g is not bounded on S.

3 (a) (25 pts) Let f be the function on $[-1,1]$ given by the formula

$$f(x) = \{ -1 -1 \le x < 0$$

5 (a) (10 pts) State the Mean Value Theorem. (b) (15 pts) Suppose that f is a function which is differentiable on (a, b) and assume that $f'(x) < 0$ for all $x \in (a, b)$. Use the Mean Value Theorem to show that f is strictly decreasing on (a, b).

6 Let f be a differentiable function on $(-\infty, \infty)$ and suppose that

$$\lim_{x\to\infty} [f(x) + f'(x)] = L \ ,$$

where L is some number (i.e., L is **not** $\pm\infty$). (a) (15 pts) Show that $\lim_{x\to\infty} f(x) = L$. (Hint: You might want to make the peculiar observation

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f(x)e^x}{e^x}$$

and calculate the limit on the right with L'Hospital's rule. If you decide to do this, make sure that you explain why it is that L'Hospital's rule applies to this situation.) (b) (10 pts) Show that $\lim_{x\to\infty} f'(x) = 0$.

7 (25 pts) Give an example of a function f on $[0, 1]$ which is **not** Darboux integrable. Explain thoroughly why your example is correct.

8 (25 pts) Give an example of a function f on $[0, 1]$ such that

"•" f is Darboux integrable, and

"•" the function $F(x)$ given by

$$F(x) = \int_0^x f(t)\, dt$$

is **not** differentiable on $(0, 1)$. Justify your answer carefully. It might be helpful to illustrate your argument by drawing the graphs of f and F.