Math336 **Exam 1** **February 20, 1998**

Instructions:

(1) The exam is 50 minutes long.

(2) SHOW ALL YOUR WORK. Explain your answers. You are more likely to get partial credit if you explain what you are doing.

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1. (20 points) Find an example of each of the following: (a) A subset A of $\mathbb{R}^3$ which is closed and unbounded.

(b) A compact set in $\mathbb{R}^2$ with no interior points.

2. (20 points) Answer true (T) or false (F) to the following statements (no proof is required):

The subset $(0,1] = \{x|0 < x \leq 1\}$ is neither closed nor open in $\mathbb{R}^1$.

The intersection of an *infinite* family F of open sets in a metric space is *always* an open subset.

The subset S of all irrational numbers in R^1 is uncountable.

The boundary of the set $A = \{(x,y)| -1 < x < 1, -1 < y < 1\}$ is
$\partial A = \{(x,y)| -1 \leq y \leq 1, |x| = 1\} \cup \{(x,y)| -1 \leq x \leq 1, |y| = 1\}$.

3. (20 points) (a) Define what it means for a subset A of a metric space S to be closed.

(b) Define what it means for a subset A of a metric space S to be compact.

4. (20 points) Let $p_n = (\frac{(-1)^n n^2}{n^2+99n+7}, 1) \in \mathbb{R}^2$, for $n = 1, 2, \ldots$. Does the sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathbb{R}^2$ have a convergent subsequence ? Explain your answer.

5. (20 points) Prove the following theorem: *"Suppose that A is a compact set in a metric space S. Then for each positive number $\delta > 0$, there is a finite number of points $p_1, p_2, \ldots, p_n$ in A such that $\cup_{i=1}^{n} B(p_i, \delta) \supset A$."*

Math336 **Exam 2** **March 27, 1998**

Instructions:

(1) The exam is 50 minutes long.

(2) SHOW ALL YOUR WORK. Explain your answers. You are more likely to get partial credit if you explain what you are doing.

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1. (20 points) Let $f : [0,1] \times [0,1] \to \mathbb{R}^1$ be the function defined by $f(x,y) = e^{x^2+y^2}$. Is f uniformly continuous on the unit square $[0,1] \times [0,1]$? Carefully justify your answer.

2. (20 points) (a) Let A be a subset of a metric space S. State what it means for a function $f : A \to \mathbb{R}^1$ to be continuous.

(b) Let $A = \{(x,y) || x| \leq \pi, |y| \leq 3\}$ and let $f : A \to \mathbb{R}$ be the function defined by $f(x,y) = e^y \sin(x)$. Is there a point $(x_0, y_0) \in A$ with $f(x_0, y_0) = \frac{1}{4}$? Carefully justify your answer.

3. (20 points) Let $A = \{(x,y) | x^2 + y^2 \leq 4\}$ and $f : A \to \mathbb{R}$ the function defined by $f(x,y) = \sin(x-y)$. Does the range of f contain its supremum? Carefully justify your answer.

4. (20 points) State and prove the following theorem: *"Let A be a compact subset of a metric space S. Suppose that $f : A \to \mathbb{R}^1$ is continuous on A. Then the range of f is bounded."*

5. (20 points) Compute the following partial derivatives: (a) $\frac{\partial f}{\partial x_2}$, where $f(x_1, x_2, x_3) = x_1^{1000} + \sin(2x_2) + x_1 x_2 x_3^2$.

(b) $\frac{\partial [f \circ g]}{\partial x_1}$, where $f(u,v) = u^2 + 2v^2$ and $g(x_1, x_2) = (2x_1 + 3x_2, x_1 + 2x_2)$.

Math336 **Exam 3 (Takehome)** **Due April 24, 1998 in class**

Instructions:

(1) This take home-exam is due April 24, 1998 in class. No late take-home exam will be accepted.

(2) This test will be conducted under the Honor Code. You should work out all problems by yourself. You are NOT allowed to give or receive any unauthorized aid on this exam.

(3) Please read and sign the following statement:

On my honor, I have abided by the code of honor and have committed no act of academic dishonesty on this examination. I have neither given nor receive any unauthorized aid on this exam.

Student's signature: ______________________________

Date: ______________________________

1. (10 points) Let $f(x,y) = 0$ and

$$f(x,y) = \frac{xy}{2x^2+y^2}, for (x,y) \neq (0,0).$$

(a) Show that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ both exist.

(b) Compute $f(x,x)$ and show that f is not continuous at $(0,0)$.

2. (15 points) Let $f(0,0) = 0$ and

$$f(x,y) = x^2 + y^2 - 2x^2 y - \frac{4x^6 y^2}{(x^2+y^2)^2},$$

for $(x,y) \neq (0,0)$. (a) Show that

$$4x^4y^2 \leq (x^2+y^2)^2$$

and $|f(x,y)| \leq |x^2+y^2-2x^2y| + x^2$. Conclude that f is continuous.

(b) Is $(0,0)$ a critical point of f ?

(c) Compute $f(x,x^2)$. Does $f(x,y)$ have a local minimum at the point $(0,0)$?

3. (10 points) Let $f(x,y) = e^{x^2+y^2}$. Find the Taylor formula of f with the remainder around the point $(0,0)$ with $n = 3$. (Hint: see Theorem 7.7 on page 185).

4. (10 points) Let $f(x,y,z) = x^2 - y^2 + z^2$, $a = (1,1,1)$ and $h = (3,4,5)$. Evaluate $\bigtriangledown f(a)$ and the directional derivative of f at $(1,1,1)$ in direction h, $< \bigtriangledown f(a), h >$.

5. (20 points) Investigate the absolute convergence or divergence of $\sum_{n=1}^{\infty} a_n$ if : (a) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

(b) $a_n = \frac{(-1)^n}{n^3}$.

6. (10 points) Find the radius of convergence of each of the following power series:

(a) $\sum_{n=1}^{\infty} n^3 x^n$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}(x-1)^n$

7. (15 points) (a) Starting with the geometric series for $\frac{1}{1-x}$, use Theorem 9.14 to show $[-\log(1-x)] = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x| < 1/2$.

(b) Evaluate $\int_0^x \log(1-t)dt$.

(c) Using the results obtained in (a), (b) and Theorem 9.14, show that

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = -x\log(1-x) + x + \log(1-x)$$

for $|x| < 1/2$.

8. (10 points) Let $f(x,y) = 3x^2 - 4xy + 3y^2 + 8x - 17y + 5$. Find the point (a,b) at which $f(x,y)$ attains its local minimum.