Math 336: Real Analysis II
Winter Semester 2000
Solutions to Exam 1

1. State the following (10 points each).

(a) The contraction mapping theorem.

(b) The definition of uniform convergence.

(c) The conditions satisfied by a metric $\rho : \mathcal{M} \times \mathcal{M} \to \mathbf{R}$ on a set $\mathcal{M}$.

2. Following are four sequences of functions $\{f_n : [0,1] \to \mathbf{R}\}$. Choose three of the four sequences, and for each one do the following. First determine whether the sequence converges pointwise and if so write down the limit function. Then determine whether the sequence also converges uniformly. (*Hint: when in doubt, pick a couple of values for n and graph.*) (5 points each)

(a) $f_n(x) = \dfrac{\sin 2\pi n x}{n}$

Solution: Converges pointwise and uniformly to 0.

(b) $f_n(x) = \begin{cases} x \text{ if } x \le 1 - 1/n \\ 0 \text{ if } x > 1 - 1/n \end{cases}$

Solution: Converges pointwise to x if $x < 1$ and to 0 if $x = 1$. Does not converge uniformly.

(c) $f_n(x) = x + (-1)^n x^2$.

Solution: Diverges.

(d) $f_n(x) = \sqrt[n]{x}$.

Solution: Converges pointwise to 1 if $x > 0$ and to 0 if $x = 0$. Does not converge uniformly.

3. Decide whether each of the following statements is true or false. If false, give a counterexample. Do four out of five parts. (5 points each)

(a) The limit of a uniformly convergent sequence of continuous functions is continuous.

Answer: True.

(b) If the limit of a uniformly convergent sequence of functions f_n is continuous, then there exists $N \in \mathbf{N}$ such that f_n is continuous when $n \geq N$.

Answer: False. Consider f_n given by $f_n(x) = 0$ if $x < 0$ and $f_n(x) = 1/n$ if $x \geq 0$. Then f_n is not continuous for any n, but f_n converges uniformly to zero (which is certainly continuous) on all of $\mathbf{R}$.

(c) A continuous function $T : \mathbf{R} \to \mathbf{R}$ with a unique fixed point is a contraction mapping.

Answer: False. Consider $T(x) = 2x$ which has the unique fixed point $x = 0$ but is not a contraction.

(d) Let $\mathcal{M}$ be the set of all continuous functions $f : \mathbf{R} \to [0, 1]$. The function $\rho(f, g) = |f(0) - g(0)|$ is a metric on $\mathcal{M}$.

Answer: False. Consider $f(x) = 0$ and $g(x) = x$. Then $\rho(f, g) = 0$ but $f \neq g$.

(e) A uniformly convergent sequence of functions also converges pointwise.

Answer: True.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function and $M \in \mathbf{R}$ be a constant such that $|f'(x)| \leq M$ for all $x \in \mathbf{R}$. Let $f_n(x) = f(x + 1/n)$. Use the Mean Value Theorem to show that $f_n(x)$ converges uniformly to $f(x)$ on $\mathbf{R}$. (15 points)

Solution: Given $\epsilon > 0$, take $N \in \mathbf{N}$ to be larger than M/ϵ. Then if $n \geq N$, we have

$$|f_n(x) - f(x)| = |f(x + 1/n) - f(x)| = \frac{|f'(c)|}{n}$$

for some $c \in [x, x + 1/n]$ by the Mean Value Theorem. Since $|f'(c)| < M$ by hypothesis, we conclude that

$$|f_n(x) - f(x)| \leq \frac{M}{n} \leq \frac{M}{N} < \epsilon.$$

□

5. Let $\mathcal{M}$ be the set of all continuous functions $f : [0,1] \to [0,1]$ (the range of f is important here). Then $\mathcal{M}$ is metric space if we declare the distance between $f, g \in \mathcal{M}$ to be $||f-g||_\infty$. Given $f \in \mathcal{M}$, define

$$Tf(x) = \frac{1}{3}\int_0^x f(s)\,ds + 1/2.$$

Show that $T : \mathcal{M} \to \mathcal{M}$ is a contraction mapping. (10 points)

Solution: Consider two functions $f, g : [0,1] \to [0,1]$. We estimate

$$\begin{aligned}\sup_{x\in[0,1]} |Tf(x) - Tg(x)| &= \sup_{x\in[0,1]} \left|\frac{1}{3}\int_0^x (f(s)-g(s))\,ds\right| \le \sup_{x\in[0,1]} \frac{1}{3}\int_0^x |f(s)-g(s)|\,ds \\ &\le \frac{1}{3}\int_0^1 |f(s)-g(s)|\,ds \le \frac{1}{3}\int_0^1 ||f-g||_\infty\,ds = \frac{1}{3}||f-g||_\infty.\end{aligned}$$

That is, $||Tf - Tg||_\infty \le \frac{1}{3}||f-g||_\infty$, so T is a contraction. □

6. Suppose that $f_n : \mathbf{R} \to \mathbf{R}$ converges uniformly to f and that $g : \mathbf{R} \to \mathbf{R}$ is uniformly continuous. Show that $g \circ f_n$ converges uniformly to $g \circ f$. (10 points)

Solution: Given $\epsilon > 0$, we know (by uniform continuity) that there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|g(x) - g(y)| < \epsilon$. Also (by uniform convergence), there exists $N \in \mathbf{N}$ such that $n \ge N$ implies $|f_n(x) - f(x)| < \delta$ for every $x \in \mathbf{R}$. Therefore $n \ge N$ implies that

$$|g \circ f_n(x) - g \circ f(x)| = |g(f_n(x)) - g(f(x))| < \epsilon$$

for every $x \in \mathbf{R}$. We conclude that $g \circ f_n$ converges uniformly to $g \circ f$ on $\mathbf{R}$. □

Extra Credit Problems.

7. In problem **??**, what function $f : [0,1] \to [0,1]$ is the fixed point of T? (5 points)

Solution: We seek a function f such that

$$f(x) = Tf(x) = \frac{1}{3}\int_0^x (f(s))^2\,ds + 1/2$$

for all $x \in [0,1]$ Note that plugging in $x = 0$ yields $f(0) = 1/2$. Differentiating both sides of the equation yields (by the fundamental theorem of Calculus)

$$f'(x) = \frac{(f(x))^2}{3}.$$

Thus f is the solution of this differential equation with the initial value $f(0) = 1/2$. You can compute the solution from here...

8. Give an example showing that the statement in problem **??** is untrue if we only assume that $g : \mathbf{R} \to \mathbf{R}$ is continuous. (5 points)

Solution: Take $f_n(x) = x + 1/n$ and $g(x) = x^2$.