Name: ________________________________ **Instructor:** Jeffrey Diller

Math 336: Real Analysis II
Spring Semester 2000
Exam 2
Friday, April 14

This Examination contains 6 problems on 7 sheets of paper including the front cover. Do all your work on the paper provided.

Scores

Question	Possible	Actual
1	15	
2	40	
3	15	
4	15	
5	15	
6	15	
Total	100	

GOOD LUCK

1. Do all three of the following (5 points each).

(a) Define what it means for a function $f : \mathbf{R}^n \to \mathbf{R}^m$ to be differentiable at a point.

(b) State the definition of an open subset of $\mathbf{R}^n$.

(c) State the Weierstrass M-test.

2. Decide whether each of the following statements is true or false. If false, give a counterexample. Do four out of five parts. (10 points each)

(a) $\sum_{j=1}^{\infty} a_j$ converges if and only if $\sum_{j=1}^{\infty} |a_j|$ converges.

(b) Let $\sum_{j=1}^{\infty} a_j$ be a series whose terms are all greater than or equal to zero and whose partial sums are all less than π. Then the series converges.

(c) Let $E_j \subset \mathbf{R}$ be closed for every $j \in \mathbf{N}$. Then $\bigcup_{j=1}^{\infty} E_j$ is closed.

(d) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function such that $\lim_{x \to 0} f(x, 0) = \lim_{y \to 0} f(0, y) = 0$. Then $\lim_{(x,y) \to (0,0)} f(x, y) = 0$.

(e) Suppose that $\sum_{j=0}^{\infty} a_j (x - 3)^j$ converges at the point $x = 1$. Then it also converges at the point $x = 4$.

3. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $f(x,y) = (\sin(x+y), \cos(x-y))$.

(a) Compute the Jacobian matrix of f at an arbitrary point (x,y) and explain why the result implies that f is differentiable everywhere in $\mathbf{R}^2$. *Do* **not** *try to use the definition of derivative in your explanation!*

(b) Suppose that $g : \mathbf{R}^2 \to \mathbf{R}^3$ is a function such that

$$Dg_{(s,t)}(\mathbf{h}) = \begin{pmatrix} s & t \\ t & s^2 \\ e^s t & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Compute the matrix for the derivative $D(g \circ f)_{(\pi/2,0)}$.

Do two out of the following three problems–15 points each

4. Consider the series $\sum_{j=1}^{\infty} \frac{1}{(x+j)^4}$. Show that this series converges uniformly for $x \in [0,1]$.

Explain why the limit function is actually C^1.

5. The Taylor series centered at 0 for $\cos x$ is $\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!}$. Show that this series converges to $\cos x$ for every $x \in \mathbf{R}$.

6. Suppose that $f, g : \mathbf{R}^n \to \mathbf{R}^n$ are C^1 functions satisfying $g \circ f(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in \mathbf{R}$. Show that if f has Jacobian matrix A at $\mathbf{x}$, then g has Jacobian matrix A^{-1} at $f(\mathbf{x})$.

Use the above assertion to show that if $f : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $f(x, y) = (x^3 - 2y^2, x \sin y)$, Then there exists no C^1 function $g : \mathbf{R}^2 \to \mathbf{R}^2$ satisfying $g \circ f(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in \mathbf{R}^2$.