Name: ________________________________ **Instructor:** Jeffrey Diller

Math 336: Real Analysis II
Spring Semester 2000
Final Exam
Wednesday, May 10

This Examination contains 8 problems on 11 sheets of paper including the front cover. Do all your work on the paper provided.

Scores

Question	Possible	Actual
1	30	
2	50	
3	25	
4	25	
5	15	
6	15	
7	15	
8	15	
Total	175	

GOOD LUCK

...and best wishes for life after real analysis.

1. Do all of the following four (10 points each).

(a) State the definition of uniform convergence $\{a_n\}_{n=1}^{\infty}$.

(b) State the contraction mapping principle

(c) State the definition of convergence for an infinite series.

(d) State the inverse function theorem

2. One of the two statements in each of the following pairs is false. Identify the false statement and give a counterexample. In lieu of a precise counterexample, you can get at least partial credit in some cases by drawing an appropriate picture. Do five of six parts. (10 points each)

(a)
- If $f_n : [0,1] \to \mathbf{R}$ converges uniformly to $f : [0,1] \to \mathbf{R}$, then f_n converges to f pointwise.
- If $f_n : [0,1] \to \mathbf{R}$ converges pointwise to $f : [0,1] \to \mathbf{R}$, then f_n converges to f uniformly.

(b)
- If $\{a_n\}$ is a sequence converging to zero, then $\sum_{n=0}^{\infty} a_n$ converges.
- If $\sum_{n=0}^{\infty} a_n$ converges, then $\{a_n\}$ is a sequence converging to zero.

(c)
- If all the partial derivatives of $f : \mathbf{R^n} \to \mathbf{R^m}$ exist at $\mathbf{0}$, then f is differentiable at $\mathbf{0}$.
- If$f : \mathbf{R^n} \to \mathbf{R^m}$ is differentiable at $\mathbf{0}$, then all the partial derivatives of f exist at $\mathbf{0}$.

(d)

- If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 function with a local inverse g defined near $f(\mathbf{x})$, then g is C^1 and $Dg_{f(\mathbf{x})} = (Df_{\mathbf{x}})^{-1}$
- If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 function and $Df_{\mathbf{x}}$ is invertible, then there is a local inverse g for f defined near $f(\mathbf{x})$.

(e) Let $\sum_{j=0}^{\infty} c_j x^j$ be a power series.

- If the the series converges at $x = R > 0$, then it converges at $x = -R$.
- If the series converges at $x = R > 0$, then it converges uniformly on $(-r, r)$ for all $r < R$.

(f) Let $f_n : \mathbf{R} \to \mathbf{R}$ be C^1 functions such that $f_n(0) = 0$ for every $n \in \mathbf{N}$.

- If f_n converges uniformly on $\mathbf{R}$, then f_n' converges uniformly on $\mathbf{R}$.
- If f_n' converges uniformly to g on $\mathbf{R}$, then f_n converges uniformly on $\mathbf{R}$.

3. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $f(x, y) = (e^x - e^y, \ln(x + y))$. Note that $f(1, 1) = (0, \log 2)$.

(a) Compute the Jacobian matrix for f (5 points).

(b) Explain why there exists a local inverse g for f defined near $(0, \log 2)$. (5 points)

(c) Give the linear approximation at $(0, \log 2)$ for g. (10 points)

(d) Using $(1,1)$ as a first guess, find a better guess at a point (x,y) such that $f(x,y) = (.1, \log 2)$ *Do* **not** *simplify your arithmetic!* (5 points).

4. Consider the intial value problem

$$y'(t) = (t + y(t))^2, \qquad y(0) = 2$$

Let $r > 0$ be given and E be the set of all continous functions $y : (-r, r) \to [0, 4]$.

(a) As in the proof of the existence and uniqueness theorem for solutions to first order ordinary differential equations, write down a a function $T : E \to E$ whose fixed point will be a solution of the initial value problem. (8 points)

(b) Given a first guess of $y(t) = 2$, use your T to find a better guess at a solution of the initial value problem. (8 points)

(c) Show that if r (see statement of problem) is chosen small enough, then T will be a contraction mapping (distance between functions is measured by the sup norm). It is important here that functions in E are bounded above and below by 4 and -4, respectively. (9 points)

Do 3 out of the following four problems. (15 points each)

5. Let $f, g : \mathbf{R}^n \to \mathbf{R}^m$ be differentiable at $\mathbf{0}$. Let $a, b \in \mathbf{R}$ be constants and $h(\mathbf{x}) = af(\mathbf{x}) + bg(\mathbf{x})$. Show that h is differentiable at $\mathbf{0}$.

6. Show that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, and $c_n = \max\{a_n, b_n\}$ for every n, then $\sum_{n=0}^{\infty} c_n$ converges.

7. Suppose that $a < b$ are numbers and $f_j : [a,b] \to \mathbf{R}$ are continuous functions converging uniformly to $f : [a,b] \to \mathbf{R}$. For each $x \in [a,b]$ and each $j \in \mathbf{N}$, let

$$F_j(x) = \int_a^x f_j(t)\,dt.$$

Show that $F_j(x)$ converges uniformly to

$$F(x) = \int_a^x f(t)\,dt.$$

8. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is an infinitely differentiable function whose derivatives satisfy

$$\left| \frac{d^n f}{dx^n} \right| \le n!$$

for every $n \in \mathbf{N}$ and every $x \in \mathbf{R}$. Let $\sum_{j=0}^{\infty} a_j x^j$ be the Taylor series centered at 0 for f. Give an interval $(-r, r)$ (i.e. find r) on which this Taylor series converges to f and prove that your answer is correct.