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Math 336: Real Analysis II
Spring Semester 2000
Exam 2
Solutions

1. Decide whether each of the following statements is true or false. If false, give a counterexample. Do four out of five parts. (10 points each)

(a) $\sum_{j=1}^{\infty} a_j$ converges if and only if $\sum_{j=1}^{\infty} |a_j|$ converges.

Solution: False. For example, $\sum_{j=1}^{\infty} \frac{(-1)^j}{j}$ converges, but $\sum_{j=1}^{\infty} \frac{1}{j}$ doesn't.

(b) Let $\sum_{j=1}^{\infty} a_j$ be a series whose terms are all greater than or equal to zero and whose partial sums are all less than π. Then the series converges.

Solution: True.

(c) Let $E_j \subset \mathbf{R}$ be closed for every $j \in$. Then $\bigcup_{j=1}^{\infty} E_j$ is closed.

Solution: False. For example, $E_j = [0, 1 - 1/j]$ is closed for every $j \in$, but $\bigcup_{j=1}^{\infty} E_j = [0, 1)$ is not closed.

(d) Let $f :^2 \rightarrow$ be a function such that $\lim_{x \to 0} f(x, 0) = \lim_{y \to 0} f(0, y) = 0$. Then $\lim_{(x,y) \to (0,0)} f(x, y) = 0$.

Solution: False. For example, the first two limits are both zero for $f(x, y) = \frac{xy}{x^2+y^2}$, but the last limit doesn't exist.

(e) Suppose that $\sum_{j=0}^{\infty} a_j (x - 3)^j$ converges at the point $x = 1$. Then it also converges at the point $x = 4$.

Solution: True.

2. Suppose that $f :^2 \rightarrow^2$ is given by $f(x, y) = (\sin(x + y), \cos(x - y))$.

(a) Compute the Jacobian matrix of f at an arbitrary point (x, y) and explain why the result implies that f is differentiable everywhere in ^{2}. *Do* **not** *try to use the definition of derivative in your explanation!*

Solution: The Jacobian matrix is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos(x+y) & \cos(x+y) \\ -\sin(x-y) & \sin(x-y) \end{pmatrix}$$

Since every entry of this matrix is defined and continuous at every point $(x, y) \in^2$, the function is differentiable on ^{2}.

(b) Suppose that $g :^2 \to^3$ is a function such that

$$Dg_{(s,t)}(\mathbf{h}) = \begin{pmatrix} s & t \\ t & s^2 \\ e^s t & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Compute the matrix for the derivative $D(g \circ f)_{(\pi/2,0)}$.

Solution:

$$\begin{aligned} D(g \circ f)_{(\pi/2,0)} &= Dg_{f(\pi/2,0)} \cdot Df_{(\pi/2,0)} = Dg_{(1,0)} \cdot Df_{(\pi/2,0)} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ e & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

3. Consider the series $\sum_{j=1}^{\infty} \frac{1}{(x+j)^4}$. Show that this series converges uniformly for $x \in [0, 1]$.

Solution: Note that $\frac{1}{(x+j)^4} \leq \frac{1}{j^4}$ for every $x \geq 0$ and that $\sum_{j=1}^{\infty} \frac{1}{j^4}$ converges. By the Weierstrass M test then, $\sum_{j=1}^{\infty} \frac{1}{(x+j)^4}$ converges uniformly on $[0, \infty)$.

Explain why the limit function is actually C^1.

Solution: If we differentiate the terms of the series, we obtain

$$\left| \frac{d}{dx} \frac{-4}{(x+j)^5} \right| \leq \frac{4}{j^5}$$

for every $j \in$. Since $\sum_{j=1}^{\infty} \frac{4}{j^5}$ converges, the M-test allows us to conclude that $\sum_{j=1}^{\infty} \frac{-4}{(x+j)^5}$ converges uniformly, too. That is, we know that the original series $f(x)$ of functions converges uniformly, and the series $g(x)$ obtained by differentiating term-by-term converges uniformly on $[0, \infty)$. Under these circumstances we have (by Theorem 6.3.3) that g is continuous and $f'(x) = g(x)$. In particular, f is C^1 on $[0, \infty)$.

4. The Taylor series centered at 0 for $\cos x$ is $\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}$ (*Note the change in the series.*) Show that this series converges to $\cos x$ for every $x \in$.

Solution: The partial sums $S_n(x)$ of the Taylor series $S(x)$ (centered at $0 \in$) for $\cos x$ are the Taylor polynomals (also centered at 0) for $\cos x$. Therefore we can apply Taylor's theorem: for each n and x, there exists a number c between 0 and x such that

$$|\cos x - S_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right|$$

But $f^{(n+1)}(c)$ will be $\pm \sin c$ or $\pm \cos c$ for every n. Hence

$$|\cos x - S(x)| = \lim_{n\to\infty} |\cos x - S_n(x)| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

To see that the last limit actually is zero, note that if $N \geq 2|x|$ and $n \geq N$, we have

$$\frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^N}{N!} \frac{|x|^{n-N+1}}{(N+1)(N+2)\dots(n+1)} \leq |x|^N \frac{1}{2^{n+1-N}} \to 0$$

as $n \to \infty$ (because x and N are fixed while n grows).

5. Suppose that $f, g :^n \to^n$ are C^1 functions satisfying $g \circ f(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in$. Show that if f has Jacobian matrix A at $\mathbf{x}$, then g has Jacobian matrix A^{-1} at $f(\mathbf{x})$.

Solution: Note that the derivative of the identity function $(\mathbf{x}) = \mathbf{x}$ is the indentity matrix I. Therefore, by the Chain Rule

$$D(g \circ f)_{\mathbf{x}} = Dg_{f(\mathbf{x})} \cdot Df_{\mathbf{x}} = I.$$

Since both matrices on the left are square $(n \times n)$, we conclude that $Dg_{f(\mathbf{x})} = (Df_{\mathbf{x}})^{-1}$.

Use the above assertion to show that if $f :^2 \to^2$ is given by $f(x, y) = (x^3 - 2y^2, x \sin y)$, Then there exists no C^1 function $g :^2 \to^2$ satisfying $g \circ f(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in^2$.

Solution: The first part of this problem implies that if g exists, then at the very least $Df_{\mathbf{x}}$ must be invertible for every $\mathbf{x} \in^2$. However,

$$Df_{(x,y)} = \begin{pmatrix} 3x^2 & \sin y \\ -4y & x \cos y \end{pmatrix},$$

so $Df_{(0,0)}$ is just the zero matrix, which is not invertible. We conclude that g does not exist.